New Zealand
Maths Olympiad Committee
Intermediate problems
Set 4

Remember that in all the following problems you are expected to provide a proof, that is, a complete and convincing argument of why your answer is correct. A simple answer, while a good start, is by no means enough!

1. Suppose that p and q are prime numbers and that the equation:

$$x^4 - px^3 + q = 0$$

has an integral root. What are the possible values of p and q?

Solution: Let r be such a root. Rearranging we get:

$$q = pr^3 - r^4 = r^3(p-1)$$

Since q is a prime, it must be the case that $r = 1$, so $q = p - 1$. Thus $p = 2$, $q = 3$, the only two consecutive primes. ■

2. Feeling a bit bored, I added up the page numbers in a book of mine. I got the answer 2003. That didn't seem right, so I looked through the book again and noticed that one of the page numbers was repeated. Assuming that my arithmetic was right after all, how many pages did the book have and which page number was repeated?

Solution: Suppose that the book had $n+1$ pages, numbered 1 to n, with page number p repeated. Then the sum of the page numbers is:

$$1 + 2 + 3 + \cdots + n + p = \frac{n(n+1)}{2} + p$$

Since $p < n + 1$ this lies strictly between the sum of the first n integers, and the sum of the first $n + 1$. Now:

$$\frac{62 \times 63}{2} < 2003 < \frac{63 \times 64}{2}$$

so $n = 62$, that is there are 63 pages. Furthermore $(62)(63)/2 = 1953$ so $p = 50$, that is, page number 50 was repeated. ■

3. In an acute angled triangle ABC, the altitude from A meets BC at A' and that from B meets AC at B'. If the length of $A'B'$ is 24cm, and the length of AB is 26 cm, what is the length of the segment connecting the midpoint of $A'B'$ to the midpoint of AB?

Solution: Since $\angle AB'B = \angle BA'A = 90°$ the points $ABA'B'$ all lie on a circle with centre the midpoint, N say, of AB. Since the radius of this circle is 13 cm (half the length of AB)

the lengths $NA' = NB' = 13cm$. So $A'B'N$ is isosceles, and the distance from the midpoint, say M of $A'B'$ to N satisfies:

$$A'M^2 + MN^2 = A'N^2.$$

So,

$$AM = \sqrt{13^2 - 12^2} = 5.$$

■

4. A function f is defined on non-negative pairs of integers as follows:

$$\begin{aligned} f(0, n) &= n + 1 \\ f(k, 0) &= f(k-1, 1) \\ f(k+1, n+1) &= f(k, f(k+1, n)). \end{aligned}$$

Determine $f(2, 1000)$.

Solution: One possibility is just to chase carefully through the definitions. A bottom up approach would seem to be less likely to lead to error (and moreover will give more information in the end). So first let's consider $f(1, k)$. From the second rule, $f(1, 0) = f(0, 1) = 2$. From the third rule

$$f(1, n+1) = f(0, f(1, n)) = f(1, n) + 1$$

So $f(1, t) = t + 2$ for any t (it starts at 2 at $t = 0$ and goes up by 1 each time we increase the second argument by 1). Now we can apply a similar argument for $f(2, t)$. Namely:

$$f(2, 0) = f(1, 1) = 3.$$

and

$$f(2, n+1) = f(1, f(2, n)) = f(2, n) + 2$$

so this time we start at 3 and go up by 2 each time. Thus

$$f(2, t) = 2t + 3.$$

So $f(2, 1000) = 2003$.

The formal name of the technique we've used here is *induction*. ■