New Zealand
Maths Olympiad Committee
Intermediate problems
Set 3

Remember that in all the following problems you are expected to provide a proof, that is, a complete and convincing argument of why your answer is correct. A simple answer, while a good start, is by no means enough!

1. A 9 cm by 12 cm rectangle is folded across its long side so that two diagonally opposite corners coincide. How long is the crease?

   Solution: This can be solved by working out various right angled triangles and using the Pythagorean theorem. A more elegant solution is to observe that the crease must form the perpendicular bisector of the diagonal. The length of the diagonal is 15 cm. If c is the length of the creases, then the triangle formed by half the crease and half the diagonal is similar to the triangle formed by the diagonal and two sides of the rectangle. So:

   $$\frac{x/2}{15/2} = \frac{9}{12}$$

   which gives $x = 45/4$ cm. ■

2. If a, b, c, d, and e are real numbers with:

   $$\begin{aligned} a+b+c+d+e &= 8 \\ a^2+b^2+c^2+d^2+e^2 &= 16 \end{aligned}$$

   find the maximum possible value for e.

   Solution: The usual method in this sort of problem is to find some quadratic expression which provides the information you want. Since we're concentrating on e it makes sense to isolate it. After some fiddling around:

   $$\begin{aligned} (8-e)^2 &= (a+b+c+d)^2 \\ &= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd. \end{aligned}$$

   Since $2xy \leq x^2+y^2$ for any real numbers x and y:

   $$\begin{aligned} (8-e)^2 &\leq 4(a^2+b^2+c^2+d^2) \\ &= 4(16-e^2). \end{aligned}$$

   This can be rearranged to give:

   $$5e^2 - 16e \leq 0$$

   or $0 \leq e \leq 16/5$.

We are not quite done as we've established an upper bound for e but haven't shown that it can be realised. In order to realise the upper bound, all the estimates we used have to be tight, that is, all the inequalities have to be equalities. That implies $a = b = c = d$. Now we can check that $e = 16/5$, $a = b = c = d = 6/5$ satisfy the equations, and so we have the proper upper bound. ■

3. x and y are two two digit numbers with $x < y$ that have the property that their four digit product begins with 2, and when the 2 is deleted, the remaining number is $x + y$. Find all such pairs of numbers.

   Solution: We are told that:

$$xy = 2000 + x + y$$

   If we moved x and y to the right, the resulting expression $xy - x - y$ would almost factor. In fact, it is $(x - 1)(y - 1) - 1$. So:

$$(x - 1)(y - 1) = 2001.$$

   Since $2001 = 3 \times 23 \times 29$, and $2000 \leq xy < 3000$, the only possibilities are $x = 24$, $y = 88$, and $x = 30$, $y = 70$. ■

4. Let n be a positive integer. How many distinct solutions are there to the equation $x+y+z = n$ where x, y, and z are non-negative integers?

   Solution: Suppose that the value of x is k. Then $y + z = n - k$. Since $0 \leq y \leq n - k$, there are $n - k + 1$ possible values for y, each leading to a unique possible value of z. So, counting from $x = 0$, the total number of solutions is:

$$(n + 1) + n + (n - 1) + \cdots + 2 + 1 = \frac{(n + 1)(n + 2)}{2}.$$

   There is a standard method of solving such problems which applies to any number of variables. In this case, start with a string of $n + 2$ marks. Choose any two marks and cross them out. This corresponds to a solution of the given equation by letting x be the number of marks to the left of the firs one crossed out (possibly 0), y the number of marks between the two crossed out (ditto), and z those to the right of the second crossed out mark. So, the number of solutions is the same as the number of ways of choosing two distinct elements from an $n + 2$ element set. This number is denoted, $\binom{n+2}{2}$ and has the value given above.

   Alternatively we could work out how to compute $\binom{n+2}{2}$ by noticing that there are $n + 2$ choices for the first mark, then $n + 1$ for the second. That seems to give $(n + 2)(n + 1)$ possibilities, but we don't care what order we choose the marks in, so we need to divide by 2, the number of orderings we could have picked the two marks in. The general formula is:

$$\binom{a}{b} = \frac{a(a - 1) \cdots (a - b + 1)}{b!} = \frac{a!}{b!(a - b)!}$$

   where $k!$ (read: k factorial) is defined to be $k(k - 1)(k - 2) \cdots (3)(2)(1)$. ■