New Zealand
Maths Olympiad Committee
Intermediate problems
Set 1

Remember that in all the following problems you are expected to provide a proof, that is, a complete and convincing argument of why your answer is correct. A simple answer, while a good start, is by no means enough!

1. Find all positive integers which are equal to 11 times the sum of their digits.

   Solution: Obviously, there are no such one digit numbers. Suppose that a two digit number with digits ab had this property. Then:

   $$10a + b = 11(a + b)$$

   but this cannot happen because the right hand side is always larger than the left hand side. Now suppose that a three digit number abc had this property. Then:

   $$100a + 10b + c = 11(a + b + c).$$

   Rearranging to make things positive we get:

   $$89a = b + 10c$$

   Since the right hand side is less than 100, we must have $a = 1$ and then get $c = 8$, $b = 9$, giving the number 198 which indeed does have this property. Write a number with four or more digits as $Abcd$ where A stands for one or more digits. Let the sum of these digits be A'. If the property held we would have:

   $$1000A + 100b + 10c + d = 11(A' + b + c + d)$$

   or

   $$1000A - 11A' + 89b = c + 10d$$

   but now the right hand side is less than 100, while the left is surely more (since $A' \leq A$ at the very least). So no numbers with four or more digits have this property, and the only number with this property is 198. ■

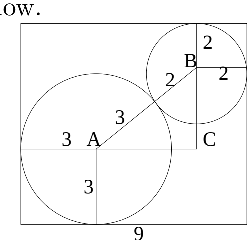
   Things to think about

   - If you could do this except for the part about four or more digits not working, that's a good start. The last part is "obvious" but sometimes it is the obvious things that are hardest to express. It's easy to rule out any particular number of digits, but to knock them all out at once is slightly trickier. An alternative would be to write the number in the form $a_k a_{k-1} \ldots a_4 bcd$, but it's still a bit messy.

- You should think about generalizations for example: Given a positive integer k is it always the case that there is a positive integer equal to k times the sum of its digits? If not, what k are possible? Is there any special pattern if we take $k = 11$, 111, 1111 etc.?

2. What is the smallest value of w so that two non-overlapping circles of diameters 4 and 6 cm can be drawn in a 9 cm by w cm rectangle? (They are allowed to touch the edge, and each other, but not to overlap properly.)

   Solution: Consider the diagram below.

   The marked lengths are correct because they represent radii of the circles involved. Since the length is 9 (all measurements in cm), $CA = 9-2-3 = 4$, and hence by Pythagoras $CB = 3$. The width of the rectangle is therefore 8.

   Oh yes, we have to argue that in the smallest such rectangle the circles fit the way we've drawn them. What are the possibilities? They might not touch the left and right sides. Then we could pull them apart slightly and then move their centres vertically towards one another making the width smaller. Likewise if the circles don't touch, move their centres vertically until they do, again decreasing the width. So, the arrangement with minimum width has the form shown, and that width is 8 cm. ■

   Things to think about

   - Some might accept the solution without the final remarks (starting "Oh yes") In fact I probably would. Still, they really should be there.
   - What's the general situation? Circles diameters a and b in a rectangle of length $c < a+b$. What's the minimum value for w? The numbers made it work out smoothly, but it shouldn't be too hard to work out a formula. In fact, just change the labels on the diagram and you'll have it.

3. Twelve people sat down around a circular table. Then they noticed that there were place-cards. Of course, no one was sitting in his/her correct spot. Show that no matter where they were supposed to sit, simply by having everyone shift left a certain number of places it can be arranged that at least *two* people are seated correctly.

   Solution: Consider how many places left each person needs to move to get to his/her correct position. This is a number from 1 to 11. There are 12 people. So two people (at least) need to move the same number of places left, which is what we want to show. ■

   Things to think about

   - So easy when you see how it's done. Problems like this often are. As a result I (and many others) find them supremely frustrating, but also supremely satisfying when they come out.

- Twelve things in eleven boxes mean at least two things in one box. That's called the *pigeonhole principle*. Remember it (only not just for 12 and 11, but also for ...). Trust me.

4. Let c be some fixed positive integer, and consider the quadratic:

$$p(x) = cx^2 + x - c$$

For which values of c (if any) are the values of this quadratic at integer values x *never* multiples of 6?

Solution: First suppose that c is not a multiple of 3. Then if x is a multiple of 3, $p(x)$ isn't (the first two terms are, but the last is not), while if x is not a multiple of 3 then x^2 leaves remainder one on division by 3, so $cx^2 - c$ is a multiple of 3 and $p(x)$ isn't. So, if c isn't a multiple of 3 then neither is $p(x)$, never mind being a multiple of 6.

Now suppose that c is a multiple of 3. If c is also a multiple of 6 then $p(x)$ is a multiple of 6 whenever x is. If, however, c is odd, then $cx^2 + x$ is always even, so $p(x)$ is always odd, and so not a multiple of 3.

To summarize, $p(x)$ is never a multiple of 6 unless c is a multiple of 6. ■

- Little facts like "the square of a non-multiple of 3 leaves remainder 1 on division by 3" and (not used here) "odd squares leave remainder 1 on division by 8" are very useful in this sort of problem.
- You'd never produce the solution in this form (I hope). You'd try lots of different values of c until you saw the pattern and *then* you would try to express it in this kind of argument. That trial and error stage is critical. What's also critical is that you be willing to throw it all away once you know what's *really* going on. A large part of becoming an effective problem solver is to know or learn the right time to switch over from just messing around mode into strongly directed behaviour. Also, sad to say, knowing when the directed behaviour isn't working out, and going back into messing around mode!
- Arguments about remainders are a lot easier if you know a notation and rules for something called *modular arithmetic* (which is just a fancy way of saying "working with remainders"). It's worth learning.