NZIMO September problems, 2001, Solutions

The solutions below to the September problems for 2001 are intended as guidelines to correct form, and to relieve any tension you may be feeling about one or more that you were unable to solve. I would like to thank everyone who participated. The problems this year were perhaps a little easier than usual, but nonetheless I was quite impressed by the quality of the submissions that I received. I am looking forward to meeting some of you at the January camp in Christchurch, and I hope to meet some of you who were perhaps not invited this year, at some future camp!

1. *How many four digit numbers are there which have the property that the sum of their digits is at most 6, and they are divisible by that sum?*

   Solution: This problem was really included as a test of organisation, and the ability to keep track of details. There were rarely any mistakes as such, but a number of incorrect answers were received because of overlooked cases, cases counted more than once, or just general sloppiness!

   The following observation will make the record-keeping a little more straightforward. Suppose that we have determined all but two digits of such a number, and that those two digits can be any pair whose sum is k. Then if the two digits include the first one, there are k possibilities, while if they do not, there are $k+1$.

   Now it is clear to proceed by working through the different digit sums one at a time. Only **1** four digit number has digit sum 1 (and of course it is divisible by 1).

   For a four digit number to be divisible by 2, its last digit must be even. For the digit sum to be 2, that last digit must be 0. Using the observation above (in a rather trivial form), there are 2 such numbers with leading digit 1, and one with leading digit 2. So there are **3** four digit numbers whose digit sum is 2 which are divisible by 2.

   Having digit sum 3 is already enough to ensure divisibility by 3. Working downwards through the cases from the first digit equal to 3 down to 1, we get:
   $$1 + 3 + (1 + 2 + 3) = 10$$
   such numbers (the term in ()'s comes from considering pairs of first two digits 12, 11, 10, and using the observation).

   For digit sum 4, the last two digits must be divisible by 4. That gives the following possibilities: 00, 20, 12. Then using the observation there are
   $$4 + 2 + 1 = 7$$
   such numbers.

For digit sum 5, the last digit must be 0. Then using the observation after having chosen the first digit to be 1 through 5 we get:

$$5 + 4 + 3 + 2 + 1 = 15$$

such numbers.

Finally, for digit sum 6, the number must be even. The last digit must be 0, 2, or 4. In each case, we can then set the first digit, and use the observation. That gives:

$$(6 + 5 + 4 + 3 + 2 + 1) + (4 + 3 + 2 + 1) + (2 + 1) = 34$$

such numbers.

Adding it all up, there are:

$$1 + 3 + 10 + 7 + 15 + 34 = 70$$

four digit numbers whose digit sum is at most six, that are divisible by their digit sum. ■

2. *A positive integer n is one more than a perfect square. Prove that $2n$ is a sum of two perfect squares.*

   Solution: This is one of those rare problems where a purely symbolic approach is both easy to find, and to understand. The given condition says that for some integer a, $n = a^2 + 1$. In that case:

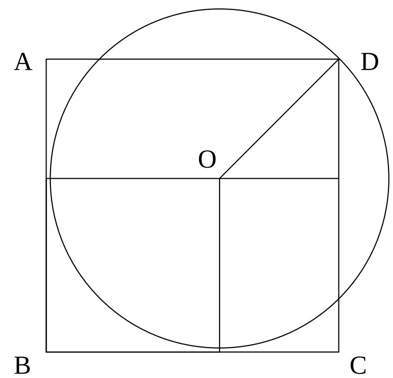
$$\begin{aligned} 2n &= 2a^2 + 2 \\ &= (a^2 + 2a + 1) + (a^2 - 2a + 1) \\ &= (a + 1)^2 + (a - 1)^2. \end{aligned}$$

   So, $2n$ is a sum of two squares.

   Of course one would probably proceed initially by looking at examples, which would tell us which two squares are to be added together to give $2n$. Once we've found that though, that work can be thrown away leaving just what remains above. ■

3. *$ABCD$ is a square of sidelength 1. A circle is tangent to two sides of $ABCD$ and passes through exactly one of its vertices. What is the radius of the circle?*

   Solution: As in most geometry problems, a diagram is crucial. Let r be the radius of the circle. In the diagram, O is the centre of the circle.



The diagram shows that r is the length of the hypotenuse OD of an isosceles right angled triangle whose legs have length $1 - r$. Therefore:

$$\begin{aligned} r^2 &= (1-r)^2 + (1-r)^2 \\ 0 &= r^2 - 4r + 2 \\ 0 &= (r-2)^2 - 2. \end{aligned}$$

Since $r - 2 < 0$ it must be the case that $r - 2 = -\sqrt{2}$, that is, $r = 2 - \sqrt{2}$.

Of course there are many many other ways to arrive at the same conclusion! There are probably even more ways of arriving at other, unfortunately incorrect, solutions. ■

4. *Among the natural numbers between 1 and 1000000 inclusive, are there more that can be written as a sum of a perfect square and a perfect cube, or are there more that cannot be so written?*

   Solution: This is a case where "rough and ready" analysis is sufficient to get the final result. The upper limit given is 10^6, so if a number in this range is the sum of a square and a cube (both positive, as most of you assumed thought it's not directly stated), then the number being squared is between 0 and $10^3 - 1$, and the one being cubed is between 0 and $10^2 - 1$. So there are only 10^3 different possibilities for the square, and 10^2 for the cube, giving at most 10^5 different sums. Since 10^5 is certainly less than $1/2$ of 10^6 there are more numbers that cannot be so written. ■

5. *A sequence $a_1, a_2, \ldots,$ of real numbers satisfies:*

$$a_1 = 2001$$

   and for every $n \geq 1$,

$$a_1 + a_2 + \cdots + a_{n-1} + a_n = n^2 a_n.$$

   Compute a_{2001}.

   Solution: Again, looking at the first few terms should expose a fairly simple pattern for the values a_n. It is then up to you to carefully check that this pattern is forced to continue. The standard tool in such cases is induction.

Alternatively, the defining recurrence can be manipulated into a simpler form, which allows expressing the values of a_n directly. I'll follow that second approach.

Note that $a_1 + \cdots + a_{n-1} = (n-1)^2 a_{n-1}$. So we have:

$$\begin{aligned} (n-1)^2 a_{n-1} + a_n &= n^2 a_n \\ (n-1)^2 a_{n-1} &= (n^2-1) a_n \\ \frac{n-1}{n+1} a_{n-1} = a_n & \end{aligned}$$

where in the last line we have used $n^2 - 1 = (n-1)(n+1)$. Unwinding this new recurrence (that is, substituting for a_{n-1} using a recurrence of the same form, and continuing in this way until we hit a_1) we get:

$$a_n = \frac{n-1}{n+1} \frac{n-2}{n} \frac{n-3}{n-1} \cdots \frac{1}{3} a_1$$

All the numerators in this product are cancelled by following denominators, except for the last two. Likewise all the denominators except the first two are cancelled by following numerators. So:

$$a_{2001} = \frac{2 \times 1}{2002 \times 2001}(2001) = \frac{1}{1001}.$$

■

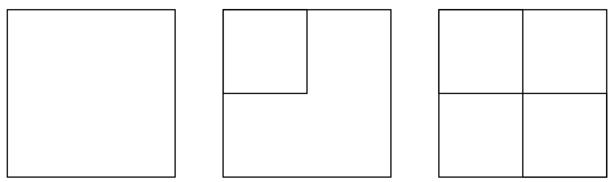
6. *You have an unlimited supply of cubic blocks of any whole number of cm per side. What is the smallest cube that you can construct using exactly 2001 of them? What is the minimum number of different sized blocks needed for constructing a cube using exactly 2001 blocks?*

   Solution: Such a cube must certainly have volume at least 2001 cc. Since $12^3 < 2001 < 13^3$, the first possible volume is 2197 cc. To use 2001 blocks we need an "excess" of 196 cc over what we would obtain using entirely 1cc block. A single 2 cm block contributes an excess of 7 cc, and it happens that $196 = 7 \times 28$. So we can construct a block 13 cm on a side by using 1973, 1 cm blocks and 28 2 cm blocks. This also answers the second part since we have only used two different block sizes, and one size alone won't work as 2001 is not a perfect cube. ■

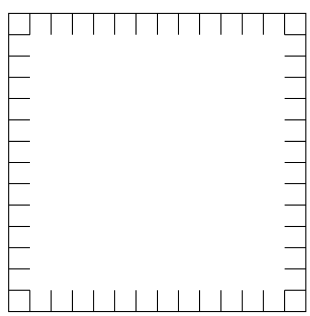
7. *Among all triangles with one side of length 5 cm, and area 6 cm^2, determine those for which the product of the length of the altitudes (measured in cm) is minimised.*

   Solution: An unintended error on my part, somehow "minimised" slipped through two rounds of proof reading where it should have been maximised. Still, it proved to be a useful exercise in checking extreme conditions. In this case, obviously one of the altitudes has length 12/5 cm. However, we can make the other two as short as we like by making the two other sides as long as we care to. Namely, draw two parallel lines a distance of 12/5 cm apart. On one of them mark out a 5 cm segment. Then take any point on the other line as far as you like from the ends of the marked segment. This forms a triangle of the required type and obviously the other two sides can be made arbitrarily long, and the corresponding altitudes arbitrarily small. So the product of the lengths of the altitudes cannot be minimised – it can be made as close as we care to 0, but of course it can never be made to equal 0 exactly. ∎

8. *A* 2×2 *grid can be drawn on a blank piece of paper by drawing only three squares as illustrated below. What is the minimum number of squares that can be drawn in order to draw a* 2001×2001 *grid?*



Solution: A trap question – and one which many fell for. The even case differs from the odd case. To give the answer first, 4000 squares suffice. To see that this many are necessary consider the segments that must be drawn pointing inwards from each side forming the boundaries of the outer ring of small squares which touch, but are not themselves part of the edge of the large square.



There are 8000 of these segments in total (2000 per side) and any single square covers at most two of them. So certainly at least 4000 squares are required. To see that 4000 squares are enough draw, for any such line the square that includes it, and includes the corner of its side from which it is furthest away. Every part of ever horizontal or vertical edge on the interior of the grid is more than half way from some edge, and is covered by one of the two squares at that height drawn to that edge (often by both of them). Also it is clear that the entire boundary of the grid is drawn in this way. (Doing the 3×3 and 5×5 cases might be helpful here). ■

9. *Maria prepares for the NZIMO selection problems by practicing for 25 days on some other problems. Each day she solves at most 10 problems. If, on any one day, she solves more than 7 problems, then she solves at most 5 problems on each of the next two days. What is the maximum number of problems that Maria can have solved over the 25 day period?*

   Solution: This was largely an exercise in careful presentation.

   Suppose that there is any day on which Maria solves more than 7 problems. If there are two days still to follow, she can solve at most 5 problems on each of those days, and hence at most 20 problems over the three day span. On the other hand she could have solved 21 problems over this span by solving 7 problems each day. So to achieve the maximum she must not solve more than 7 problems on any days except possibly the 24th and the 25th. Again if she solves more than 7 problems on the 24th, she can solve at most 15 over the two days. On the other hand she could instead solve 17 by solving 7 on the 24th and 10 on the 25th day. So, the maximum number of problems that she can have solved is:

$$24 \times 7 + 10 = 178.$$

■

10. *Let ABC be a triangle and P a point inside it such that:*

$$\angle PBC = \angle PCA < \angle PAB.$$

   Suppose that PB cuts the circumcircle of ABC at B and E and that CE cuts the circumcircle of APE at E and F.

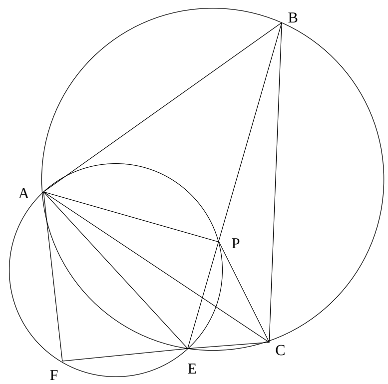
   (a) *Show that $Area(APE) = Area(AEC)$.*

   (b) *Show that*

$$\frac{Area(APEF)}{Area(ABP)} = \left(\frac{AC}{AB}\right)^2.$$

   Solution:

   Consider the diagram below:



Observe first that the condition that $\angle PCA < \angle PAB$ ensures that F lies beyond E, not between C and E. Now, $\angle PCA = \angle PBC = \angle EBC$ (given) and $\angle EBC = \angle EAC$ (angles on the same arc), so $\angle EAC = \angle PCA$ and thus $AE \parallel CP$. So triangles APE and ACE have equal areas. Now note that:

$$\begin{aligned} \text{Area}(APEF) &= \text{Area}(AEF) + \text{Area}(APE) \\ &= \text{Area}(AEF) + \text{Area}(ACE) \\ &= \text{Area}(ACF) \end{aligned}$$

Also angle APE is complementary to both $\angle AFE$ and $\angle APB$ so the latter two are equal as are angles ACE and ABE. Thus the triangles ACF and ABP are similar, and the desired equality simply expresses the fact that the ratio of the area of similar figures equals the square of the ratio of their corresponding sides.

■

11. *Let a, b, and c be real numbers, and $f(x) = ax^2 + bx + c$. Suppose that:*

$$\begin{aligned} |f(-1)| &\leq 1 \\ |f(0)| &\leq 1, \textit{ and} \\ |f(1)| &\leq 1. \end{aligned}$$

Prove that for all x with $-1 \leq x \leq 1$,

$$|f(x)| \leq 5/4.$$

Solution: This appeared to be the most difficult problem of the set. There is a serious temptation to make unjustified claims about how the function "must" equal 1 at some point, or its maximum "must" occur in a certain range, or that the "worst case" is ... The words "worst case" or "best case" in a proof are an almost certain tip off that the writer has a feeling for how the argument should go, but hasn't been able to plug, or hasn't spotted a hole in the method. Just a word to the wise, now on with the proof.

I will though give a proof in that spirit (just to show that it can be done!) Let a function f satisfying the conditions be given. First note that if $a = 0$, so that the function is linear or constant, then in fact $|f(x)| \leq 1$ for $-1 \leq x \leq 1$ (since this is true at the endpoints). So we may certainly suppose that $a \neq 0$. Also, by reflecting the graph of f in the x-axis if necessary we may assume that $a > 0$. Finally, by reflecting in the y-axis if necessary we may assume that $b \leq 0$, that is, the vertex of the parabola occurs at 0 or at some positive value. So we may rewrite the form of f as:

$$f(x) = a(x - \alpha^2) + d$$

for some $a > 0$ and $0 \leq \alpha$. Our goal is to show that $d \geq -5/4$. To this end, we note that if $f(-1) = f(1)$ (i.e. $\alpha = 0$) then $d = f(0)$ so there is no problem. So we may assume that $\alpha > 0$ which means that $-1 \leq f(1) < f(-1) \leq 1$. Now if $f(0) > -1$ we could translate the graph of f slightly to the left without violating the given inequalities. Since this also has the effect of *decreasing* $f(-1)$ we could then expand the scale on the y-axis slightly, thus producing a quadratic g with:

$$g(x) = a'(x - \beta)^2 + d'$$

still satisfying the inequalities, but with $d' < d$. So, we may assume that $f(0) = -1$, or in other words that:

$$f(x) = a(x - \alpha)^2 - a\alpha^2 - 1.$$

Then from $f(1) \geq -1$ we get:

$$a(1 - 2\alpha) \geq 0$$

that is $\alpha \leq 1/2$. On the other hand from $f(1) \leq 1$ we get:

$$a(1 + 2\alpha) \leq 2.$$

Adding the non-positive quantity $a(2\alpha - 1)$ to this gives:

$$4a\alpha \leq 2 \quad \text{or} \quad a\alpha \leq 1/2.$$

Therefore $a\alpha^2 = (a\alpha)\alpha \leq 1/4$ and so

$$d = -a\alpha^2 - 1 \geq -5/4.$$

Therefore f is bounded in absolute value below $5/4$ on $[-1, 1]$. ∎

12. *Inania is an odd country. Every word in their language uses only the letters i, n, or a (and all three must occur), is exactly six letters long, and contains an even number of a's. Subject to these restrictions, every possible combination of letters is a word.*

(a) *How many words are there in the language?*

Solution: This is simply a matter of counting by cases. There must be two or four a's. Having chosen the positions of the a's, the remaining places can be filled arbitrarily with i's or n's, except for the two arrangements consisting of entirely one letter. So the total number of words is:

$$\binom{6}{2}(2^4 - 2) + \binom{6}{4}2^2 - 2 = 15 \times 14 + 15 \times 2 = 240.$$

■

(b) *Owing to a shortage of words, you have been commissioned by the government of Inania to determine how many words would be available if they changed the "six letter rule" (and only that rule) to an "n letter rule" for some positive integer n. Please fulfill your commission.*

Solution: Using the argument above we could get this answer as a certain sum involving binomial coefficients and powers of 2. A more concrete answer is expected in problems of this type so we could then use various identities about the binomial coefficients, and simplify the answer. I'm going to sketch a different approachw which uses the technique of *generating functions*. To avoid notation clash, let the number of letters we're trying to use be t. Let a, i, and n simply be variable symbols. Consider first the expansion of:

$$(a + i + n)^t.$$

Each term in this expansion represents a "word" obtained from choosing either an a, an i, or an n, in each of the t factors. Put another way, there is an exact correspondence between the terms of the expression above when it is expanded fully, and unrestricted t letter words using only the letters a, i, or n. We don't want to allow there to be an odd number of a's, so we would like to cancel all the terms where a occurs an odd number of times. This is easily done in the expression:

$$f(a, i, n) = \frac{(a + i + n)^t + (-a + i + n)^t}{2}.$$

(By including $-a$, all the terms where the exponent of a, that is, the number of times that a occurs in the corresponding word, is odd, will cancel – we need to divide by 2, because the terms where the exponent is even get doubled.) That's still not good enough, as there are terms which involve only two of the three letters, and indeed only one of the three letters remaining in the expansion. However, the first group of these correspond to $f(0, i, n)$, $f(a, 0, n)$, and $f(a, i, 0)$, so we will subtract those terms off. In doing so, each word consisting of a single letter alone is subtracted twice, so must be added back in (we only wanted to subtract it once). What we wind up with is that there is an exact correspondence between the terms of:

$$f(a,i,n)-f(a,i,0)-f(a,0,n)-f(0,i,n)+f(a,0,0)+f(0,i,0)+f(0,0,n)$$

and the words that we are trying to count. How do we change "counting terms" to a number? By making sure that each of the terms has value 1. Again, this is easy to accomplish – finally set the variables all to equal 1. So the number we want is:

$$f(1,1,1)-f(1,1,0)-f(1,0,1)-f(0,1,1)+f(1,0,0)+f(0,1,0)+f(0,0,1)$$

which is:

$$\frac{3^t+1-2^t-2^t-2(2^t)+1+(-1)^t+2+2}{2} = (1/2)(3^t-2^{t+2}+6+(-1)^t).$$

Just to check our work, substitute $t=6$:

$$(1/2)(729-256+6+1) = (1/2)(480) = 240.$$

■