NZIMO September problems, 2002

The following problems are intended to help us select the 24 students who will be asked to attend a camp in Christchurch in January 2003, from among whom we will select the team of six members who will represent New Zealand at the International Mathematical Olympiad in Tokyo, in July 2003.

Olympiad problems are very difficult and *always* require solutions or proofs – a calculation alone, however accurate, is never worth more than 1 point out of 7. Although these problems are much much easier than those which you might see at an Olympiad, the same marking scheme prevails. That is, you must in each case provide a solution, a clear, logical, and complete explanation of why your answer is the only possible correct one. Sometimes you are asked to give a proof explicitly, but even when you are not one is implicitly required.

The questions are arranged roughly in order of difficulty – but as different people have different problem solving strengths, you may find some of the later problems easier than earlier ones. Please note that a few correct solutions will earn much more credit than a large number of fragmentary ones. Your time will generally be better spent making sure that your solutions to problems that you know how to do are complete, than in struggling with problems which you do not understand.

It is hard to predict how many problems you will need to solve in order to obtain an invitation to the camp. If you have been able to complete more than half the problems (and remember – I mean truly complete) you should feel that you have made a very good effort indeed.

Please in order to make the job of grading the problems easier, follow the following instructions carefully:

- Put your name on *each* piece of paper that you are submitting.
- Staple or clip your papers together (but not with the registration form) in the upper left hand corner only.
- It is much easier if you begin each solution on a fresh piece of paper, or at least a fresh side.

Thanks very much for your participation, and good luck with the problems!

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1. In the year 2002, Susan will become as old as the sum of the digits of her year of birth. In what year was she born?

2. In triangle ABC, D and E lie on AB and AC respectively, and the line DE meets BC produced in F. If $AD = 1$, $DB = CE = 2$, $EA = 3$ and $BC = 4$, determine CF.

3. Determine all triples (a, b, c) of positive integers such that:
$$\frac{1}{a+2} + \frac{1}{b+2} = \frac{1}{2} + \frac{1}{c+2}.$$

4. What is the smallest number of $+$ signs in the expression
$$+1 + 2 + 3 + 4 + \cdots + 99 + 100$$
that can be changed to $-$ signs so that the value of the expression is 2002?

5. Bonnie and Chris share a pizza. The chef divides it into four pieces with two straight cuts at right angles to one another. Bonnie chooses the first piece and then the pieces are taken alternately in clockwise order. How should Bonnie choose her first piece in order to ensure that she gets at least as much of the pizza as Chris? What if the division procedure is the same, but the chef makes three cuts, through a single point, at angles of $60°$ to one another?

6. If x and y are nonzero numbers satisfying $x^2 + xy + y^2 = 0$, find the value of:
$$\left(\frac{x}{x+y}\right)^{2002} + \left(\frac{y}{x+y}\right)^{2002}.$$

7. Find the last two digits of the number $2^{2002}(2^{2003} - 1)$ when it is written in base 10.

8. A quadrilateral $ABCD$ is given where $AD = BC$ and $\angle CDA > \angle BCD$. Prove that $AC > BD$.

9. A mathematical olympiad took place over two days, each day having the same number of problems. It was noted that the number of problems solved by each participant on the first and the second day differed by one. On the other hand, for each problem number, the number of participants who solved that problem on the first and the second day differed by two. Show that the number of participants must have been even.

10. Show that there is exactly one increasing sequence of nonnegative integers $a_0, a_1, a_2, \ldots$ which has the property that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j$ where i and j are not necessarily distinct. Also evaluate a_{2002}.