NZIMO September problems, 2001

The following problems are intended to help us select the 24 students who will be asked to attend a camp in Christchurch in January 2002, from among whom we will select the team of six members who will represent New Zealand at the International Mathematical Olympiad in Glasgow, in July 2002.

Olympiad problems are very difficult and *always* require solutions or proofs – a calculation alone, however accurate, is never worth more than 1 point out of 7. Although these problems are much much easier than those which you might see at an Olympiad, the same marking scheme prevails. That is, you must in each case provide a solution, a clear, logical, and complete explanation of why your answer is the only possible correct one. Sometimes you are asked to give a proof explicitly, but even when you are not (such as in questions 3 and 4) one is implicitly required. That is, in question 3 it is certainly not enough to make a scale diagram of the situation and carefully measure the radius of the circle – you must find some exact value for this radius, and a proof that it is correct. In question 4 it is not enough to guess the answer based perhaps on a consideration of some smaller cases – you must determine the correct answer and argue logically for its correctness.

The questions are arranged roughly in order of difficulty – but as different people have different problem solving strengths, you may find some of the later problems easier than earlier ones. Please note that a few correct solutions will earn much more credit than a large number of fragmentary ones. Your time will generally be better spent making sure that your solutions to problems that you know how to do are complete, than in struggling with problems which you do not understand.

It is hard to predict how many problems you will need to solve in order to obtain an invitation to the camp. If you have been able to complete more than half the problems (and remember – I mean truly complete) you should feel that you have made a very good effort indeed.

Please in order to make the job of grading the problems easier, follow the following instructions carefully:

- Put your name on *each* piece of paper that you are submitting.
- Staple or clip your papers together (but not with the registration form) in the upper left hand corner only.
- It is much easier if you begin each solution on a fresh piece of paper, or at least a fresh side.

Thanks very much for your participation, and good luck with the problems!

Sample Problem and Solution from 2000.

The sum of three nonnegative real numbers x_1*,* x_2*, and* x_3 *is not greater than* $1/2$*. Prove that:*

$$(1 - x_1)(1 - x_2)(1 - x_3) \geq \frac{1}{2}.$$

Non-solution

Since the product on the left gets smaller as the x's get bigger, we want the sum to equal $1/2$. Also the sum of the factors on the right is then fixed, and products of things with fixed sums get smaller as the factors get further apart. To make them as far apart as possible, make one of the x's equal $1/2$ and the others 0. That makes the product $1/2$, and any other arrangement makes it bigger.

Comments

The conclusion is correct, but the steps to get there are unclear and unjustified. This could be used as the "skeleton" of a solution, but a direct approach is probably easier.

Solution

Certainly each x is less than or equal to $1/2$, and so the three factors on the left are positive. Expanding the left hand side, and then using simple estimates we get:

$$\begin{aligned}
(1 - x_1)(1 - x_2)(1 - x_3) &= 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3 \\
&= 1 - (x_1 + x_2 + x_3) + x_1x_3 + x_2x_3 + x_1x_2(1 - x_3) \\
&\geq 1 - (1/2) + 0+) + 0 \\
&= 1/2.
\end{aligned}$$

In the inequality we have simply used the given information about the sum of the x's, the fact that each is non-negative, and the non-negativity of $1 - x_3$.

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1. How many four digit numbers are there which have the property that the sum of their digits is at most 6, and they are divisible by that sum?

2. A positive integer n is one more than a perfect square. Prove that $2n$ is a sum of two perfect squares.

3. $ABCD$ is a square of sidelength 1. A circle is tangent to two sides of $ABCD$ and passes through exactly one of its vertices. What is the radius of the circle?

4. Among the natural numbers between 1 and 1000000 inclusive, are there more that can be written as a sum of a perfect square and a perfect cube, or are there more that cannot be so written?

5. A sequence a_1, a_2, $\ldots$, of real numbers satisfies:

$$a_1 = 2001$$

and for every $n \geq 1$,

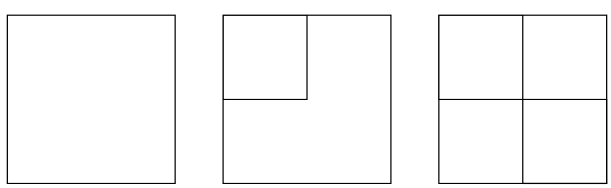
$$a_1 + a_2 + \cdots + a_{n-1} + a_n = n^2 a_n.$$

Compute a_{2001}.

6. You have an unlimited supply of cubic blocks of any whole number of cm per side. What is the smallest cube that you can construct using exactly 2001 of them? What is the minimum number of different sized blocks needed for constructing a cube using exactly 2001 blocks?

7. Among all triangles with one side of length 5 cm, and area $6\,\text{cm}^2$, determine those for which the product of the length of the altitudes (measured in cm) is minimised.

8. A 2×2 grid can be drawn on a blank piece of paper by drawing only three squares as illustrated below. What is the minimum number of squares that can be drawn in order to draw a 2001×2001 grid?



9. Maria prepares for the NZIMO selection problems by practicing for 25 days on some other problems. Each day she solves at most 10 problems. If, on any one day, she solves more than 7 problems, then she solves at most 5 problems on each of the next two days. What is the maximum number of problems that Maria can have solved over the 25 day period?

10. Let ABC be a triangle and P a point inside it such that:

$$\angle PBC = \angle PCA < \angle PAB.$$

Suppose that PB cuts the circumcircle of ABC at B and E and that CE cuts the circumcircle of APE at E and F.

(a) Show that $\text{Area}(APE) = \text{Area}(AEC)$.

(b) Show that

$$\frac{\text{Area}(APEF)}{\text{Area}(ABP)} = \left(\frac{AC}{AB}\right)^2.$$

11. Let a, b, and c be real numbers, and $f(x) = ax^2 + bx + c$. Suppose that:

$$\begin{aligned} |f(-1)| &\leq 1 \\ |f(0)| &\leq 1, \text{ and} \\ |f(1)| &\leq 1. \end{aligned}$$

Prove that for all x with $-1 \leq x \leq 1$,

$$|f(x)| \leq 5/4.$$

12. Inania is an odd country. Every word in their language uses only the letters i, n, or a (and all three must occur), is exactly six letters long, and contains an even number of a's. Subject to these restrictions, every possible combination of letters is a word.

(a) How many words are there in the language?

(b) Owing to a shortage of words, you have been commissioned by the government of Inania to determine how many words would be available if they changed the "six letter rule" (and only that rule) to an "n letter rule" for some positive integer n. Please fulfill your commission.