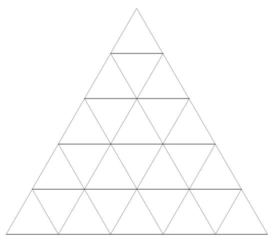
New Zealand Mathematical Olympiad 2004

Problem Set 3

1. 2004 ones are written around a circle. Two players alternate making moves. A legal move is to erase two neigboring mumbers and replace them with their sum. A winner is the player who managed to write 4. If at the end only one number remains, different from 4, the game ends in a draw. Who has a winning strategy, the first player or the second?

2. Each side of equilateral triangle is of length n and is divided into n equal segments of unit lengths by $n-1$ points. Through these points lines parallel to the sides of the triangle are drawn which divide the equlateral triangle into a number of equal equilateral triangles (the picture below shows an example for $n = 5$)



   What is the maximal number of segments of unit length with endpoints at vertices of those triangles can be painted red so that there is no triangle whose all three sides are red?

3. Points E and F are chosen on the sides AB and CD of the convex quadrilateral $ABCD$. Prove that the midpoints of the segments AF, BF, CE, ED are also vertices of a convex quadrilateral whose area does not depend on the choice of E and F.

4. Prove that, for all positive real numbers a, b, c,

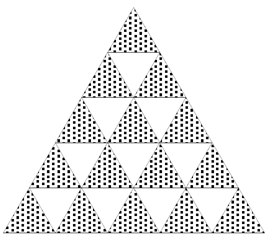
   (a) $a^3 + b^3 \geq ab(a + b)$,

   (b) $\dfrac{1}{a^3 + b^3 + abc} + \dfrac{1}{b^3 + c^3 + abc} + \dfrac{1}{c^3 + a^3 + abc} \leq \dfrac{1}{abc}$.

Solutions

1. The second player has winning strategy. Let us note that when 3 appears for the first time, the game ends after next move. Indeed, if 3 appears, then it replaces numbers 1 and 2 in some order, say 1, 2. It is not possible to have another 2 so that we have 1,2,2 standing together because the player making the move will not replace 1 and 2 with 3 but will win replacing 2 and 2 with 4. Hence we have the sequence 1,2,1 and after the move we will get 3 and 1 standing together allowing the next player to win. Let us prove now that it is the first player that will write 3. Indeed, the strategy of the second player is to give a symmetric reply unless an opportunity to win arises. If after a move of the first player it is not possible to win, hence no pairs 1,3 or 3,1 or 2,2, then a symmetric reply will not create such pairs too. So the second player cannot win.

2. The total number of unit segments is $3n(n+1)/2$. If we colour all segments which are not parallel to the base of the triangle, we will colour $n(n+1)$ segment and no red triangles will be formed. On the other hand, let us shade some of the the triangles as shown below



   In order to prevent shaded triangles to be red we can paint red only two of the three sides of each shaded triangle. Since there are $n(n+1)/2$ such triangle and no two of them share a side, we can paint no more than $2n(n+1)/2 = n(n+1)$ segment.

3. Denote N, L, M, K the midpoints of the segments AF, BF, CE, ED, respectively. The segments LN and KM join the midpoints of AF, BF and CE, DE, respectively. Hence their lengths are half the length AB and CD and the angle between them is the angle between AB and CD, call it α. Both LN and KM pass through the midpoint O of EF, hence intersect. Thus $KLMN$ is convex (otherwise its diagonals would not intersect). His area is $S = \frac{1}{2}KL \cdot MN \cdot \sin \angle KON = \frac{1}{2}KL \cdot MN \cdot \sin\alpha$, and does not depend on E and F.

4. The obvious inequality $(a-b)(a^2-b^2) \geq 0$ implies $a^3+b^3 \geq ab(a+b)$, so

$$\frac{1}{a^3+b^3+abc} \leq \frac{1}{ab(a+b)+abc} = \frac{c}{abc(a+b+c)}.$$

Similarly,

$$\frac{1}{b^3+c^3+abc} \leq \frac{1}{bc(b+c)+abc} = \frac{a}{abc(a+b+c)}.$$

and

$$\frac{1}{c^3+a^3+abc} \leq \frac{1}{ca(c+a)+abc} = \frac{b}{abc(a+b+c)}.$$

Adding them all together we get the inequality.