NZIMO 2003 Wednesday Test
Solutions and Comments

1. *For any positive integer n let*

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}.$$

Compute

$$f(1) + f(2) + \cdots + f(40).$$

Solution: When we rationalize the denominator (and remember $4n^2 - 1 = (2n-1)(2n+1)$) we obtain

$$f(n) = \frac{1}{2}\left(\sqrt{(2n+1)^3} - \sqrt{(2n-1)^3}\right)$$

and so the sum simply telescopes, becoming:

$$(1/2)\left(\sqrt{81^3} - 1\right) = 364.$$

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2. *Let $a_1, a_2, \ldots, a_n, \cdots$ be an infinite arithmetic progression of positive integers. Show that it contains an infinite subset all of whose elements have the same sum of their decimal digits.*

Solution: If the arithmetic progression is constant, the result is trivial. So, suppose that it is not constant and let its common difference be d. Let 10^k be a power of 10 larger than the first element of the progression and larger than d. Then, for all $n > k$ there is at least one element of the sequence whose value lies between 10^n and $10^n + 10^k$. The digit sum of such an element lies between 1 and $1 + 9k$. In particular, there are only a fixed number of values it can take. Since there are infinitely many such elements, one of the values of the digit sums must occur infinitely often as required. ∎

3. *For n an odd positive integers, colour the squares of an $n \times n$ chessboard black and white with all four corners being black. A* tromino *is an L-shape formed by three connected squares. For which values of n is it possible to cover all the black squares with non-overlapping trominoes (which are all contained within the board). When it is possible, what is the minimum number of trominoes needed?*

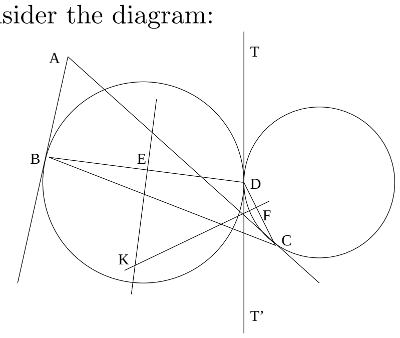
Solution: Write $n = m + 1$. Consider the black squares which are an even height above the bottom row (this includes those in the bottom row). There are $(m+1)^2$ such squares, no two of which can be covered by a single tromino. So, if a covering is possible, at least $(m+1)^2$ trominoes are required.

This rules out $n = 1, 3, 5$ since in each of those cases $3(m+1)^2$ is bigger than n^2. For $n = 7$, $3(m+1)^2 = 48$, so if a covering exists it must use the exact minimum number of trominoes required that is 16. It is not hard to accomplish this. One way is to tile the four 3×4 rectangles around the centre black square. This misses a black square, but this can be fixed by taking one of the adjacent trominoes which is covering only one black square and moving it.

Now carry on inductively. Having covered the black squares of a $(2m-1) \times (2m-1)$ board with m^2 trominoes, make a $(2m+1) \times (2m+1)$ board by surrounding it with 2×2 squares ($2m-3$ of them), and two 2×3 rectangles. These can be covered with an additional $2m-3+4 = 2m+1$ trominoes, which is the required number. ∎

4. *Let S_1 and S_2 be two circles tangent at a point D. Let A be a point on neither S_1 nor S_2. A tangent from A to S_2 meets S_1 in two distinct points, and then touches S_2 at C. A tangent from A to S_1 touches S_1 at a point B, which is on the opposite side of the line AC from D. Prove that the circumcentre of BCD lies on the circumcircle of ABC.*

Solution: Consider the diagram:



Here K is the centre of the circumcircle of BCD (E and F are such that KE and KF are perpendicular bisectors of BD and CD respectively), and TDT' is the common tangent of the two circles.

We aim to prove that $\angle BKC = 180^\circ - \angle BAC$.

Since the tangents at the end of a chord are commonly inclined to the chord we have: $\angle TDB = \angle ABD$ and $\angle T'DC = \angle DCA$. Hence:

$$\begin{aligned} \angle BDC &= \angle BDT' + \angle T'DC \\ &= 180^\circ - \angle ABD + \angle DCA \\ &= 180^\circ - (\angle ABC - \angle DBC) + (\angle DCB - \angle ACB) \\ &= (180^\circ - \angle ABC - \angle ACB) + (\angle DBC + \angle DCB) \\ &= \angle BAC + 180^\circ - \angle BDC. \end{aligned}$$

Thus

$$2\angle BDC = 180^\circ + \angle BAC.$$

Finally,

$$\begin{aligned} \angle BKC &= \angle BKD + \angle DKC \\ &= 2\angle EKD + 2\angle DKF = 2\angle EKF \\ &= 2(180^\circ - \angle BDC) = 180^\circ - \angle BAC. \end{aligned}$$

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