NZIMO 2003 Wednesday Test

1. For any positive integer n let
$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}.$$
Compute
$$f(1) + f(2) + \cdots + f(40).$$

2. Let $a_1, a_2, \ldots, a_n, \ldots$ be an infinite arithmetic progression of positive integers. Show that it contains an infinite subset all of whose elements have the same sum of their decimal digits.

3. For n an odd positive integers, colour the squares of an $n \times n$ chessboard black and white with all four corners being black. A *tromino* is an L-shape formed by three connected squares. For which values of n is it possible to cover all the black squares with non-overlapping trominoes (which are all contained within the board). When it is possible, what is the minimum number of trominoes needed?

4. Let S_1 and S_2 be two circles tangent at a point D. Let A be a point on neither S_1 nor S_2. A tangent from A to S_2 meets S_1 in two distinct points, and then touches S_2 at C. A tangent from A to S_1 touches S_1 at a point B, which is on the opposite side of the line AC from D. Prove that the circumcentre of BCD lies on the circumcircle of ABC.