New Zealand
Maths Olympiad Committee
September problems
2003

The following problems are intended to help us select the 24 students who will be asked to attend a camp in Christchurch in January 2004, from among whom we will select the team of six members who will represent New Zealand at the International Mathematical Olympiad in Athens, in July 2004.

Olympiad problems are very difficult and *always* require solutions or proofs – a calculation alone, however accurate, is never worth more than 1 point out of 7. Although these problems are much much easier than those which you might see at an Olympiad, the same marking scheme prevails. That is, you must in each case provide a solution, a clear, logical, and complete explanation of why your answer is the only possible correct one. Sometimes you are asked to give a proof explicitly, but even when you are not one is implicitly required.

The questions are arranged roughly in order of difficulty – but as different people have different problem solving strengths, you may find some of the later problems easier than earlier ones. Please note that a few correct solutions will earn much more credit than a large number of fragmentary ones. Your time will generally be better spent making sure that your solutions to problems that you know how to do are complete, than in struggling with problems which you do not understand.

It is hard to predict how many problems you will need to solve in order to obtain an invitation to the camp. If you have been able to complete more than half the problems (and remember – I mean truly complete) you should feel that you have made a very good effort indeed.

Please in order to make the job of grading the problems easier, follow the following instructions carefully:

- Put your name on *each* piece of paper that you are submitting.
- Staple or clip your papers together (but not with the registration form) in the upper left hand corner only.
- It is much easier if you begin each solution on a fresh piece of paper, or at least a fresh side.

Thanks very much for your participation, and good luck with the problems!

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1. Six people sit around a circular table. In how many different ways can three pairs of people shake hands simultaneously so that no arms cross?

2. A semi-circular region with straight edge AB of length 10 is given. What is the area of the largest rectangle that can be drawn in this region with one edge on AB?

3. Are there two 2003 digit numbers A and B with the following properties: the digits of B are a rearrangement of those of A, and the digits of $A + B$ are all 9's? Is there such a pair of numbers with 2004 digits?

4. For which real values of a does the following system of equations have real solutions?
$$\begin{aligned} |x| + |y| &= 1 \\ x^2 + y^2 &= a. \end{aligned}$$

5. Find all positive real numbers x such that:
$$x + \lfloor \frac{x}{6} \rfloor = \lfloor \frac{x}{2} \rfloor + \lfloor \frac{2x}{3} \rfloor.$$
(Note $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.)

6. What is the maximum number of three element subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$, which have the property that any two of the subsets have at most one element in common?

7. A convex quadrilateral $ABCD$ is inscribed in the circle with diameter AB. Let S be the intersection point of AC and BD and T the foot of the perpendicular from S to AB. Prove that ST bisects angle CTD.

8. The sequence of positive integers $(1, 1, 1, 1, 2)$ has the property that the sum of any three numbers in the sequence is divisible by the sum of the other two. Is there a sequence of five *distinct* positive integers with this property?

9. Show that there are no triples of positive integers (a, b, c) such that:
$$(2a + b)(2b + a) = 2^c.$$

10. Consider the following solitaire puzzle. The board is a circular ring of holes which may contain pegs. A move consists of removing a peg from some hole and then changing the state of its two adjacent holes (that is, for each one, if it contained a peg take it away, if it was empty, add a peg). The starting position has just a single peg in one of the holes and the goal is to reach the position where there are no pegs in the holes. Is this possible if the board consists of 12 holes? What about 10 holes? In general?