NZIMO 2003 Camp Problems III
Solutions and Comments

1. *Suppose that a, b, c, x, y, and z are real numbers such that $ax^3 = by^3 = cz^3$ and $(1/x) + (1/y) + (1/z) = 1$. Prove that:*

$$\sqrt[3]{ax^2 + by^2 + cz^2} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

Solution: Let $A = ax^3 = by^3 = cz^3$. Then $ax^2 = A/x$, $by^2 = A/y$, $cz^2 = A/z$ and so

$$\sqrt[3]{ax^2 + by^2 + cz^2} = \sqrt[3]{A((1/x) + (1/y) + (1/z))} = \sqrt[3]{A}.$$

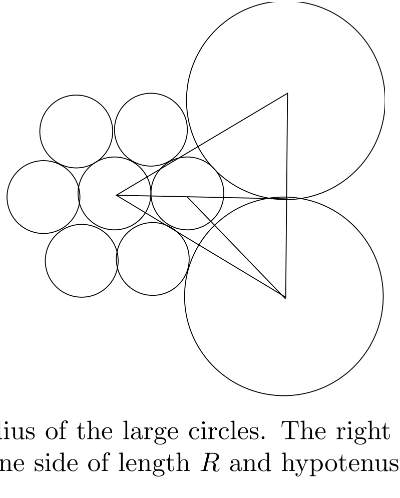
On the other hand $a = A/x^3$ and so $\sqrt[3]{a} = \sqrt[3]{A}/x$ (and similarly for b and c) gives on the right hand side:

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{A}((1/x) + (1/y) + (1/z)) = \sqrt[3]{A}.$$

On inspecting the proof we see that given the first equality among the variables but not necessarily the second, there is a more general relationship between the expression on the left and on the right. ■

2. *A circle C of radius 1 is surrounded by six other circles, each of the same radius and mutually tangent to C and its two neighbours. These in turn are surrounded by six circles, each tangent to two of the six previous circles, and to its two neighbours. What is the radius of the circles in the largest ring?*

Solution: Consider the diagram:



Let R be the radius of the large circles. The right angled triangle at the lower right has one side of length R and hypotenuse of length $R + 1$. So, its horizontal side has length $\sqrt{2R+1}$. So, the equilateral triangle with sidelength $2R$ has altitude $\sqrt{2R+1}+2$. Thus we obtain:

$$R\sqrt{3} = \sqrt{2R+1}+2.$$

Rearranging and squaring gives:

$$3R^2 - 4\sqrt{3}R + 4 = 2R + 1$$

or

$$3R^2 - 2(1+2\sqrt{3})R + 3 = 0.$$

Which yields (since $R > 1$ determines the sign to choose in the quadratic formula)

$$R = \frac{1+2\sqrt{3}+2\sqrt{1+\sqrt{3}}}{3}.$$

■

3. *Let a_1, a_2, …, a_n, … be an increasing sequence of positive integers. An element of the sequence is called good if it can be written as a sum of (not necessarily distinct) other elements of the sequence. Prove that, for some N, all the a_n with $n \geq N$ are good.*

   Solution: Let $a = a_1$. Consider the congruence classes of all the remaining members of the sequence modulo a. Since there are only a possible classes, we can choose N such that for all $n \geq N$, $a_n \equiv a_k \pmod{a}$ for some $k < N$. But then a_n equals a_k plus some number of copies of a_1. ■

4. *Find all functions f from the whole numbers to the whole numbers with the property that:*

   $$f(3x+2y) = f(x)f(y)$$

   for all whole numbers x and y.

   Solution: From $x = y = 0$ we get $f(0) = f(0)^2$, so $f(0) = 0$, or $f(0) = 1$.

   Consider the first of these cases. By setting $x = 0$ we get $f(2y) = 0$ for all y, and by setting $y = 0$ we get $f(3x) = 0$ for all x. Let $f(1) = a$. Then from $x = y = 1$, $f(5) = a^2$, and from $x = y = 5$, $f(25) = a^4$. But $f(25) = f(3 \cdot 3 + 2 \cdot 8) = f(3)f(8) = 0$. Therefore, $a = 0$. Since any $k > 4$ can be written as $3x + 2y$ for some whole numbers x and y it now follows easily by induction that $f(k) = 0$ for all k.

   Now consider the second case, $f(0) = 1$. Then for all x (using $y = 0$, $f(3x) = f(x)$, and for all y, $f(2y) = f(y)$. Let $f(1) = a$. As above, $f(25) = a^4$. But also $f(25) = f(3)f(8) = a^2$, so $a = 0$ or $a = 1$. The first option leads to $f(k) = 0$ for all $k > 0$ as before, while the second leads similarly to $f(k) = 1$.

   In summary there are three such functions: identically 0, identically 1, or equal to 1 at 0 and 0 for all positive arguments. ■