NZIMO 2003 Camp Problems III

1. Suppose that a, b, c, x, y, and z are real numbers such that $ax^3 = by^3 = cz^3$ and $(1/x) + (1/y) + (1/z) = 1$. Prove that:

$$\sqrt[3]{ax^2 + by^2 + cz^2} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

2. A circle C of radius 1 is surrounded by six other circles, each of the same radius and mutually tangent to C and its two neighbours. These in turn are surrounded by six circles, each tangent to two of the six previous circles, and to its two neighbours. What is the radius of the circles in the largest ring?

3. Let $a_1, a_2, \ldots, a_n, \ldots$ be an increasing sequence of positive integers. An element of the sequence is called good if it can be written as a sum of (not necessarily distinct) other elements of the sequence. Prove that, for some N, all the a_n with $n \geq N$ are good.

4. Find all functions f from the whole numbers to the whole numbers with the property that:

$$f(3x + 2y) = f(x)f(y)$$

for all whole numbers x and y.