NZIMO 2003 Camp Problems II
Solutions and Comments

1. *Suppose that n is a product of four distinct primes a, b, c, and d such that:*

$$\begin{aligned} a + c &= d \\ a(a + b + c + d) &= c(d - b) \\ 1 + bc + d &= bd. \end{aligned}$$

   Determine n.

   Solution: From the first equation either a or c must be 2. Suppose $c = 2$. Then the left hand side of the third equation is even, while the right is odd. Therefore $a = 2$, and $d = c + 2$. Substituting in the third equation gives $d = 2b - 1$ (and so $c = 2b - 3$). Substituting all this in the second equation gives:

$$2(2 + b + 2b - 3 + 2b - 1) = (2b - 3)(b - 1)$$

   Rearranging:

$$2b^2 - 15b + 7 = 0,$$

   or $(2b - 1)(b - 7) = 0$. So $b = 7$ (1/2 is not a prime!), $d = 13$, $c = 11$, $a = 2$, $n = 2 \times 7 \times 11 \times 13 = 2002$. Obviously a problem left over from last year! ∎

2. *A $3 \times n$ grid is filled as follows. The first row consists of the numbers 1 through n in ascending order. The second row is of the form:*

$$i,\ (i + 1),\ \dots,\ n,\ 1,\ \dots,\ (i - 1)$$

   for some $1 \leq i \leq n$. The third row has the numbers from 1 through n in some order, subject to the condition that the sum of the entries in each column of the grid is the same. For which values of n is this possible, and for such values, in how many ways is it possible?

   Solution: Let s be the sum in each column. Since the sum in each row is $n(n + 1)/2$ we get:

$$ns = \frac{3n(n + 1)}{2}.$$

   Since this gives $s = 3(n + 1)/2$, n must be odd, say $n = 2m + 1$ and $s = 3(m + 1)$.

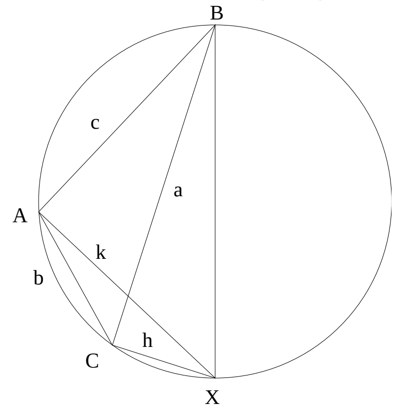
   Suppose that the 1 in row 2 occurs in column j. Then the third entry of that column must be $s - j - 1 = 3(m + 1) - j - 1 = 3m + 2 - j$. Therefore $3m + 2 - j \leq 2m + 1$ which gives $j \geq m + 1$. In the previous column the entries are $j - 1$, $2m + 1$ and $m + 3 - j$. So $m + 3 - j \geq 1$ and $j \leq m + 2$. So the 1 entry in the second row must be in column $m + 1$ or $m + 2$.

   It is then easy to check that both these possibilities can occur, so it is always possible to fill the grid in precisely two ways. ∎

3. *A triangle has sides of lengths a, b, and c, and its circumcircle has radius R. Prove that the triangle is right-angled if and only if* $a^2+b^2+c^2 = 8R^2$.

Solution: If the triangle is right-angled, then the longest side, c say is the diameter of the circumcircle. So $c = 2R$ and $a^2 + b^2 = c^2$ so $a^2 + b^2 + c^2 = 2(2R)^2 = 8R^2$.

For the converse, consider the following diagram:



Here BX is a diameter of the circumcircle, and we have labelled the lengths AX and CX as h and k respectively. If either $h = 0$ or $k = 0$ then one of the sides of the triangle coincides with a diameter of its circumcircle, so the triangle is right angled as required. So suppose that neither h nor k is 0. Since BCX and ABX are right angled:

$$\begin{aligned} a^2 + k^2 &= (2R)^2 \\ c^2 + h^2 &= (2R)^2. \end{aligned}$$

Therefore, $a^2+c^2+h^2+k^2 = 8R^2$. But $a^2+b^2+c^2 = 8R^2$, so $h^2+k^2 = b^2$, so AXC is right angled, with right angle at X. As the angle subtended by AC is right, AC must be a diameter of the circle (contrary to appearances!) and so ABC is a right angled triangle. ■

4. *Let* $0 < a, b, c < 1$. *Prove that:*

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}$$

and determine when equality occurs.

Solution: Consider the function

$$f(x) = \frac{x}{1-x}.$$

Then it is easy to establish that f is convex and strictly increasing on $[0, 1)$. Hence:

$$f(\sqrt[3]{abc}) \leq f(\,(a+b+c)/3\,) \leq \frac{f(a)+f(b)+f(c)}{3}.$$

That is:

$$\frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}} \leq (1/3)\left(\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}\right)$$

which is obviously equivalent to what we want. Furthermore equality can occur only if $a = b = c$ (otherwise the first inequality in the sequence of inequalities for f is strict), and this is clearly also sufficient.

A second solution uses AM-GM more directly. First note:

$$\begin{aligned}\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} &\geq 3\sqrt[3]{\frac{abc}{(1-a)(1-b)(1-c)}} \\ &= \frac{3\sqrt[3]{abc}}{\sqrt[3]{(1-a)(1-b)(1-c)}}.\end{aligned}$$

So if we could establish that:

$$\sqrt[3]{(1-a)(1-b)(1-c)} \leq 1 - \sqrt[3]{abc}$$

we'd be done. But again directly by AM-GM:

$$\sqrt[3]{(1-a)(1-b)(1-c)} + \sqrt[3]{abc} \leq \frac{(1-a)+(1-b)+(1-c)}{3} + \frac{a+b+c}{3} = 1$$

which is what we want, and again equality requires $a = b = c$. ∎