NZIMO 2003 Camp Problems II

1. Suppose that n is a product of four distinct primes a, b, c, and d such that:

$$\begin{aligned} a + c &= d \\ a(a + b + c + d) &= c(d - b) \\ 1 + bc + d &= bd. \end{aligned}$$

   Determine n.

2. A $3 \times n$ grid is filled as follows. The first row consists of the numbers 1 through n in ascending order. The second row is of the form:

$$i,\ (i + 1),\ \ldots,\ n,\ 1,\ \ldots,\ (i - 1)$$

   for some $1 \leq i \leq n$. The third row has the numbers from 1 through n in some order, subject to the condition that the sum of the entries in each column of the grid is the same. For which values of n is this possible, and for such values, in how many ways is it possible?

3. A triangle has sides of lengths a, b, and c, and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

4. Let $0 < a, b, c < 1$. Prove that:

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1 - \sqrt[3]{abc}}$$

   and determine when equality occurs.