New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2010 — Instructions

Solutions due date: 27th August 2010

These problems will be used by the NZMOC to select students for its International Mathematical Olympiad Training Camp, to be held in Christchurch between the 9th and 15th of January 2011. Only students who attend this camp are eligible for selection to represent New Zealand at the 2011 International Mathematical Olympiad (IMO), to be held in the Netherlands in July 2011. The cost of the camp is $460, which includes domestic airfares etc. as some sponsorship is available.

At the camp a squad of 10–12 students will be chosen for further training, and to take part in several international competitions, including the Australian and Asia-Pacific Mathematical Olympiads. The New Zealand team for the 2011 IMO will be chosen from this squad.

There are two sets of problems: junior division and senior division.

- If you are currently in year 12, **or** you have been a member of the NZIMO training squad, then you may **only** attempt the senior problems.
- If you are currently in year 11 or below, **and** you have never been a member of the NZIMO training squad, then you may attempt **both** sets of problems (and your results from both sets will be taken into account in the selection process).

Note: Students in year 11 or below who have previously attended a January camp, but not been a member of the NZIMO training squad, are no longer required to do just the senior problems.

General instructions:

- Although some problems seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided. In fact, an answer alone will only be worth 1 point out of 7.
- You may not use a calculator, computer or the internet to assist you in solving the problems.
- All solutions must be entirely your own work.
- We do not expect many, if any, perfect submissions. So, please submit all the solutions and partial solutions that you can find.

Students submitting solutions should be intending to remain in school in 2011 and should also hold New Zealand Passports or have New Zealand Resident status. To be eligible for the 2011 IMO you must have been born on or after 20 July 1991, and must not be formally enrolled in a University or similar institution prior to the IMO.

Your solutions, together with a completed Registration Form (overleaf), should be sent to

> Professor Ivan Reilly, Department of Mathematics, University of Auckland, Private Bag 92019, Auckland Mail Centre, Auckland 1142

arriving no later than 27th August 2010. You will be notified whether or not you have been selected for the Camp by mid October.

If you have any questions, please contact Ivan Reilly (i.reilly@auckland.ac.nz, (09) 923 8786) or Chris Tuffley (c.tuffley@massey.ac.nz, (06) 350 9099 x3573).

July 2010
www.mathsolympiad.org.nz

Registration Form
NZMOC Camp Selection Problems 2010

Name: ____________________

Gender: male/female

School year level in 2010: ____________________

Home address: ____________________

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Email address: ____________________

Home phone number: ____________________

School: ____________________

School address: ____________________

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Principal: ____________________

HOD Mathematics: ____________________

Do you intend to take part in the camp selection problems for any other Olympiad camp? yes/no

If so, and if selected, which camp would you prefer to attend? ____________________

Have you put your name forward for a Science camp or any other camp in January? yes/no

Are there any criminal charges, or pending criminal charges against you? yes/no

Some conditions are attached to camp selection. You must be:

- Born on or after 20 July 1991
- Studying in 2011 at a recognised secondary school in NZ
- Available in July 2011 to represent NZ overseas as part of the NZIMO team if selected.
- A NZ citizen or hold NZ resident status.

Declaration: I satisfy these requirements, have worked on the questions without assistance from anyone else, and have read, understood and followed the instructions for the January camp selection problems. I agree to being contacted through the email address I have supplied.

Signature: ____________________ Date: ____________________

Attach this registration form to your solutions, and send them to

> Professor Ivan Reilly, Department of Mathematics, University of Auckland, Private Bag 92019, Auckland Mail Centre, Auckland 1142,

arriving no later than 27th August 2010.

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Camp Selection Problems 2010

Due: 27th August 2010

Junior division

J1. We number both the rows and the columns of an 8×8 chessboard with the numbers 1 to 8. Some grains of rice are placed on each square, in such a way that the number of grains on each square is equal to the product of the row and column numbers of the square. How many grains of rice are there on the entire chessboard?

J2. AB is a chord of length 6 in a circle of radius 5 and centre O. A square is inscribed in the sector OAB with two vertices on the circumference and two sides parallel to AB. Find the area of the square.

J3. Find all positive integers n such that $n^5 + n + 1$ is prime.

J4. Find all positive integer solutions (a, b) to the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{n}{\operatorname{lcm}(a,b)} = \frac{1}{\gcd(a,b)}$$

for

(i) $n = 2007$;

(ii) $n = 2010$.

Note: "$\operatorname{lcm}(a, b)$" means the least common multiple of a and b, and "$\gcd(a, b)$" means the greatest common divisor of a and b. For example, $\operatorname{lcm}(18, 30) = 90$, and $\gcd(18, 30) = 6$.

J5. The diagonals of quadrilateral $ABCD$ intersect in point E. Given that $|AB| = |CE|$, $|BE| = |AD|$, and $\angle AED = \angle BAD$, determine the ratio $|BC|/|AD|$.

J6. At a strange party, each person knew exactly 22 others.

For any pair of people X and Y who knew one another, there was no other person at the party that they both knew.

For any pair of people X and Y who did not know each other, there were exactly six other people that they both knew.

How many people were at the party?

Senior division

S1. For any two positive real numbers $x_0 > 0$, $x_1 > 0$, a sequence of real numbers is defined recursively by

$$x_{n+1} = \frac{4\max\{x_n, 4\}}{x_{n-1}} \qquad \text{for } n \geq 1.$$

Find x_{2010}.

Note: "$\max\{x, y\}$" means the maximum of x and y — that is, whichever of the two numbers x and y is the larger. For example, $\max\{2, 3\} = 3$.

S2. In a convex pentagon $ABCDE$ the areas of the triangles ABC, ABD, ACD and ADE are all equal to the same value x. What is the area of the triangle BCE?

S3. Let p be a prime number. Find all pairs (x, y) of positive integers such that

$$x^3 + y^3 - 3xy = p - 1.$$

S4. A line drawn from the vertex A of an equilateral triangle ABC meets the side BC at D and the circumcircle at P. Show that

$$\frac{1}{|PD|} = \frac{1}{|PB|} + \frac{1}{|PC|}.$$

S5. Determine the values of the positive integer n for which

$$A = \sqrt{\frac{9n-1}{n+7}}$$

is rational.

S6. Suppose a_1, a_2, ..., a_8 are eight distinct integers from $\{1, 2, \ldots, 16, 17\}$. Show that there is an integer $k > 0$ such that there are at least three different (not necessarily disjoint) pairs (i, j) such that $a_i - a_j = k$.

Also find a set of seven distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that there is no integer $k > 0$ with that property.

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