New Zealand Mathematical Olympiad Committee

2010 April Problems — Hints

1. *A coin has been placed at each vertex of a regular 2008-gon. These coins may be rearranged using the following move: two coins may be chosen, and each moved to an adjacent vertex, subject to the requirement that one must be moved clockwise and the other anti-clockwise. Decide whether, using this move, it is possible to rearrange the coins into*

   (a) 8 heaps of 251 coins each;

   (b) 251 heaps of 8 coins each.

   Hint: After some experimenting, you've probably come to suspect that (a) is possible, but (b) is not. The trick to showing that (b) really is impossible is to construct an *invariant*: something that doesn't change as the coins are moved around.

   To do this, number the vertices from 1 to 2008, assign value i to any coin currently at vertex i, and let S be the sum of the values of the coins. What happens to S as you move the coins? (Make sure you don't miss looking at any special cases here.) What is the initial value of S? What would be true of the value, if you succeeded in forming 251 piles of eight coins?

2. *Let $ABCD$ be a convex quadrilateral that is not a parallelogram. The straight line passing through the midpoints of the diagonals of $ABCD$ intersects the sides AB and CD in the points M and N respectively. Prove that the triangles ABN and CDM have the same area.*

   Hint: The line MN divides each of the triangles ABN and CDM in two. Can you find any relationships among the areas of the resulting four triangles?

3. *Find all positive integers m and n such that $6^m + 2^n + 2$ is the square of an integer.*

   Hint: What can you say about the left-hand side if m, n are both at least 2? Can such a number be a square? Now look at the cases $m = 1$ and $n = 1$ separately.

4. *Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying*

   $$\frac{f(x+y)+f(x)}{2x+f(y)} = \frac{2y+f(x)}{f(x+y)+f(y)},$$

   for all $x, y \in \mathbb{N}$.

   Hint: What happens if $x = y$? This should tell you something about $f(n)$ when n is *even*. Now investigate what happens when n is odd, by looking at the possibilities x odd, $y = 1$, and x even, $y = 1$. Don't forget to check that the function you find actually satisfies the given equation.

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