Solutions

1. Let $n = 403$. Then Johnny saved

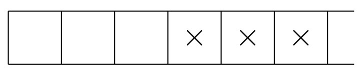
$$n^5 - (n-1)^2(n^3 + 2n^2 + 3n + 4) =$$
$$n^5 - (n^5 + 2n^4 + 3n^3 + 4n^2 - 2n^4 - 4n^3 - 6n^2 - 8n + n^3 + 2n^2 + 3n + 4) = 5n - 4.$$

As $5n - 4 = 2011$, Johnny has enough money to buy a playstation.

2. Yes, it is possible since $5^2 + 12^2 = 13^2$.

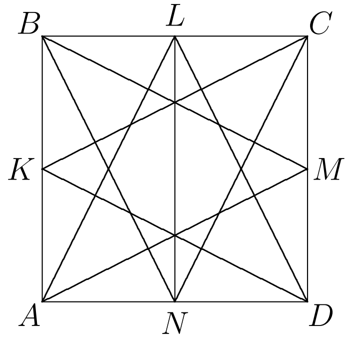
3. Obviously x_0 is also a root of the new quadratic. Therefore it has another real root x (which may coincide with x_0 though). By Vieta's theorem $x_0 + x = -\frac{1}{2}(a_1 + a_2) = \frac{1}{2}(x_0 + x_1 + x_0 + x_2)$. Hence $x = \frac{1}{2}(x_1 + x_2)$.

4. Answer: the second player wins. In his first move the second player colours three squares close to one of the ends of the strip (left end on the picture below) in such a way as to leave only three uncoloured squares at that end.



Let us call those three uncloured squares "the reserve." The first player cannot make his move colouring the squares of the reserve, he must play elsewhere. The strategy of the second player is to make any moves outside the reserve until it is no longer possible. If it is not possible for the second player to make a move outside the reserve, then the first player cannot make any move at all. At this moment the second player colours the squares of the reserve and wins.

5. Let $ABCD$ be the given quadrilateral and K, L, M, N be the midpoints of AB, BC, CD, DA, respectively. Suppose that CK is the only one among eight line segments which is not equal to a.



Then in isosceles triangles ALD and BNC the segment LN is the median and hence the altitude. Further, the right-angled triangles ALN and BNL are equal, hence $AN = BL$ and $ANLB$ is a rectangle. Similarly, $DNLC$ is a rectangle and hence $ABCD$ is a rectangle too. This now implies that right-angled triangles DKA and CKB are equal with $CK = KD$. This contradict our assumption that $CK \neq a$.

6. We may assume that $2b = a + c$. Then we may have one of these:

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}, \quad \frac{2}{a} = \frac{1}{b} + \frac{1}{c}, \quad \frac{2}{c} = \frac{1}{a} + \frac{1}{b}.$$

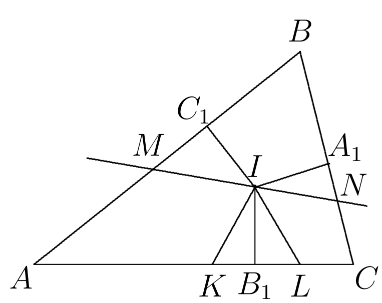
In the first case, we have $2ac = ab+cb$ from which we obtain $4ac = (a+c)^2$ or $(a-c)^2$, i.e. we obtain $a = b = c$ and the numbers are not distinct. In the second case we have $2bc = ab + ac$ or $(a + c)c = a(a + c)/2 + ac$. This implies $(a/c)^2 + a/c - 2 = 0$ from which $a/c = 1$ or $a/c = -2$. In the second case we have $a = -2c$, $b = -c/2$. It is better to express the numbers in terms of b: we have $c = -2b$ and $a = 4b$. The integer b can be arbitrary. For example, we have $-2, 1, 4$ and their reciprocals $-1/2, 1/4, 1$.

7. Substituting $x = y$ we get $f(x) = x^2 + a/2$, where $a = f(0)$. Now substituting $x = y = 0$ we get $a = 2a$, hence $a = 0$. Hence $f(x) = x^2$ is the only function that may satisfy the equation. Now we have to check that it indeed satisfies the equation: $(x - y)^2 = x^2 + y^2 - 2xy$ is indeed true. So the answer is: $f(x) = x^2$.

8. Firstly, since a and b have the same sum of digits, they have the same remainders on division by 9, hence $a - b$ must be divisible by 9, i.e. the sum of digits of $a - b$ must be divisible by 9. However this sum of digits is n, so $n \geq 9$. This value of n can be realised, for example, as follows:

$$9012345678 - 8901234567 = 111111111.$$

9. Let us drop perpendiculars IC_1, IA_1, IB_1 from I onto AB, BC, CA, respectively.



These perpendiculars are equal in length. Moreover, $AC_1 = AB_1$ (as the two tangents to the incircle drawn from the same point) and similarly $CA_1 = CB_1$. The right-angled triangles IKB_1 and INA_1 are equal since since $IB_1 = IA_1$ and $\angle IKB_1 = \angle INA_1$. Thus we have $B_1K = A_1N$. Similarly, $B_1L = C_1M$. Thus $AM + KL + CN = AM + MC_1 + NA_1 + CN = AC_1 + CA_1 = AB_1 + CB_1 = AC$.

10. The first player writes 5 in the upper left corner and divides the remaining numbers into pairs: $(1, 8)$, $(2, 7)$, $(3, 6)$, $(4, 9)$. Let us call these pairs selected. He also divides the table without the left upper corner square into four 1×2 dominoes A, B, C, D which contain squares a_1, a_2, b_1, b_2, c_1, c_2 and d_1, d_2, respectively.

5	d_1	d_2
a_1	b_1	c_1
a_2	b_2	c_2

If the second player writes a number in one of the dominoes, then the first player finds a selected pair with this number, takes the other number of this pair and writes it down in the same domino. For example, if the second player writes 4 in d_2, then the first player writes 9 in d_1. Let us explain why this strategy works. In three of the four selected pairs the sum of numbers is 9, let us call them good pairs, and only one pair is bad. At the end of the game there will be a good pair either in domino A or in domino D which will make either the sum in the first row or the sum in the first column equal to 14.