New Zealand Mathematical Olympiad Committee

Collinearity and Concurrence

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1 Introduction

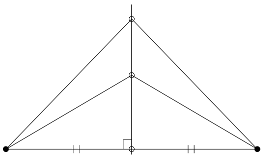
These notes cover a selection of techniques for proving collinearity and concurrence.

Why treat the two notions simultaneously? A deep answer is that the two concepts are *dual*: correct reasoning about lines and points can, sometimes, be inverted to give equally correct reasoning about points and lines. In these notes, I have deliberately avoided making these ideas explicit – that's a whole other can of worms!

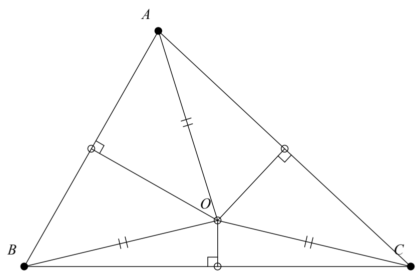
A simpler answer is that most collinearity results can be rephrased as results about concurrence, and conversely. Many of the results in these notes come in pairs.

2 Symmetric properties

Example 1 (Existence of circumcentre)**.** The three perpendicular bisectors of the three sides of a triangle ABC are concurrent. For the perpendicular bisector l_c of the side AB of a triangle ABC is the set of all points equidistant from A and B.



The analogous statements hold for l_a and l_b. Since AB and AC are not parallel, the lines l_c and l_b are not parallel, and so they meet at some point O equidistant from A, B and C. The point O must therefore lie on l_c as well.



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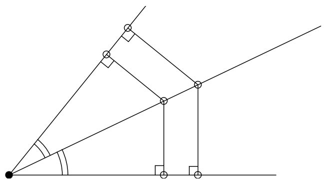
Example 2 (Existence of incentre)**.** The three internal angle bisectors of the angles at the three vertices of a triangle ABC are concurrent. For the angle bisector l_A of the internal angle $\angle A$ of a triangle ABC is the set of all points inside that sector equidistant from the sides AB and AC.



The analogous statements hold for l_B and l_C. We can conclude as in Example 1. □

Exercise 1 (Existence of A-excentre)**.** Let ABC be a triangle. Show that the internal angle bisector of $\angle A$ and the external angle bisectors of $\angle B$ and $\angle C$ are concurrent.

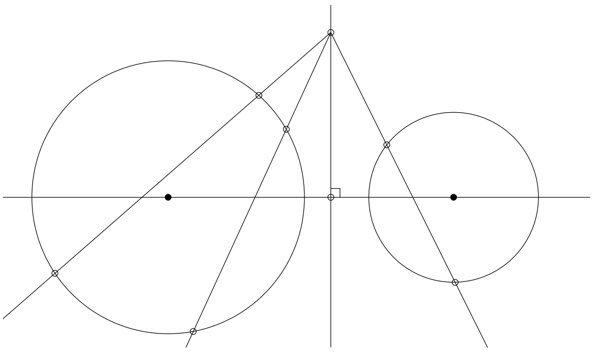
Let ω_1 and ω_2 be two circles with radii r_1 and r_2 respectively, and distinct (!) centres O_1 and O_2 respectively separated by a distance $O_1O_2 = d > 0$. Then we define the *radical axis* of ω_1 and ω_2 to be the perpendicular to O_1O_2 through X, where X is the point on the line O_1O_2 for which, considering directed ratios, we have

$$O_1X : XO_2 = d^2 + r_1^2 - r_2^2 : d^2 - r_1^2 + r_2^2.$$

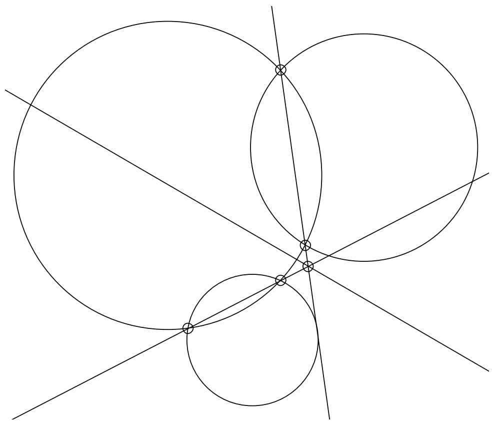
$$\text{The point } X \text{ is on the } \begin{array}{ll} \text{ray } \overrightarrow{O_2O_1} \text{ beyond } O_1, & \text{if } d^2 < r_2^2 - r_1^2; \\ \text{point } O_1, & \text{if } d^2 = r_2^2 - r_1^2; \\ \text{segment } O_1O_2, & \text{if } d^2 > |r_1^2 - r_2^2|; \\ \text{point } O_2, & \text{if } d^2 = r_1^2 - r_2^2; \\ \text{ray } \overrightarrow{O_1O_2} \text{ beyond } O_2, & \text{if } d^2 < r_1^2 - r_2^2. \end{array}$$

In particular, if ω_1 and ω_2 intersect at distinct points A and B, then the radical axis of ω_1 and ω_2 is the line AB.

Example 3 (Radical axes theorem)**.** The three radical axes of any three circles ω_1, ω_2, ω_3 with non-collinear centres are concurrent. For the radical axis l_{12} of ω_1 and ω_2 is the set of all points whose powers with respect to these two circles are equal.



The analogous statements hold for the radical axes l_{13} and l_{23}. Since O_1, O_2 and O_3 are not collinear, O_1O_2 and O_1O_3 are not parallel, and so l_{12} and l_{13} are not parallel either. Let P be their point of intersection. Then P has equal powers with respect to all three circles, and so it lies on l_{23} also.



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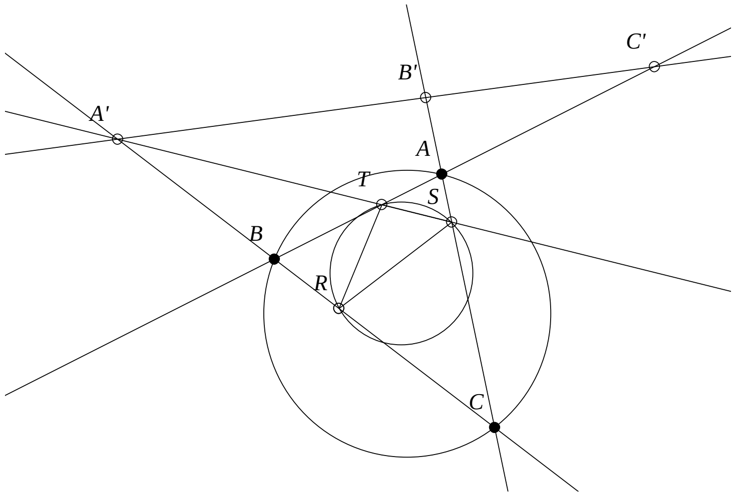
Exercise 2 (Existence of orthocentre)**.** Let R, S and T be the feet of the altitudes from A, B and C respectively of a triangle ABC. Show that $ABRS$, $BCST$ and $CATR$ are all cyclic. Deduce that ABC's altitudes AR, BS and CT are concurrent.

Example 4 (Existence of orthic axis)**.** Let R, S and T be the feet of the altitudes from A, B and C respectively of a scalene triangle ABC. Then the three intersection points A' of BC and ST, B' of CA and TR and C' of AB and RS are collinear.

For the quadrilateral $BTSC$ is cyclic; the line BC is the radical axis of ABC's circumcircle and the circumcircle of $BTSC$, and the line ST is the radical axis of the nine-point circle and the circumcircle of $BTSC$. So A' lies on the radical axis of ABC's circumcircle and nine-point circle. The same holds for B' and C'.



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The P-*Apollonius circle* of two points A and B, not equidistant from P, is the circumcircle of P, X and Y, where X and Y are the two points on the line AB for which

$$\frac{AX}{BX} = \frac{AY}{BY} = \frac{AP}{BP}.$$

Exercise 3 (Existence of first isodynamic point)**.** Show that the P-Apollonius circle of two points A and B, not equidistant from P, is precisely the set of all points Q in the plane PAB for which $AQ/BQ = AP/BP$. Deduce that in a scalene triangle ABC, the A-, B- and C-Apollonius circles of BC, CA and AB respectively meet at a point.

3 Homotheties

Two sets of points Σ and Σ' in one-to-one correspondence are *homothetic*, if the following two conditions hold:

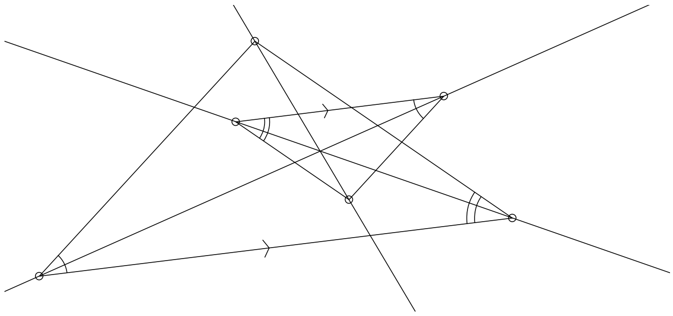
(i) Σ and Σ' are *similar* (that is, distances between corresponding points are in a fixed ratio).

(ii) Σ and Σ' are *similarly situated* (that is, lines connecting corresponding points are parallel).

Theorem 1 (Existence of centre of homothety)**.** *If Σ and Σ' are homothetic sets of points, then then lines connecting corresponding pairs of points are either all concurrent or all parallel. If concurrent, we call the point of concurrence the* centre of homothety *of Σ and Σ'.*

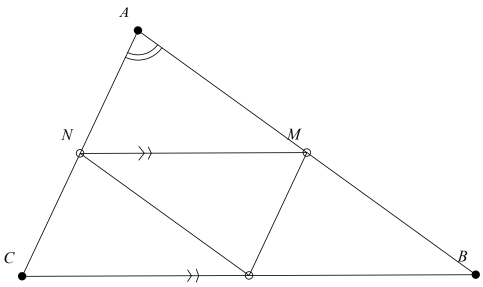
To use this concurrence result, you need to prove that two sets of points are homothetic. For example, you could show that:

1. some two corresponding "base subsets" Π and Π' of Σ and Σ', from which all the other points are constructed, are similar; and,
2. some corresponding segments AB and $A'B'$ of Σ and Σ' are parallel.



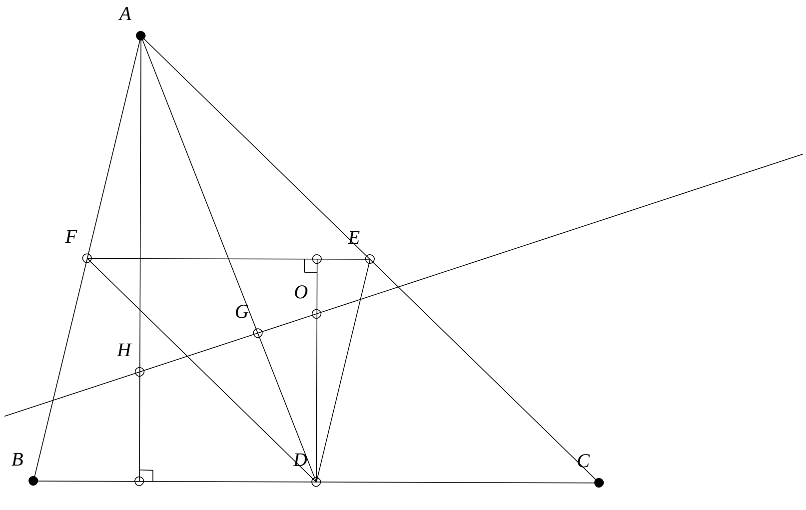
With triangles, and homothetic sets of points constructed from triangles, a fast way of demonstrating homothety is therefore to show that the corresponding sides of the three triangles are parallel.

Example 5 (Existence of centroid)**.** The three medians of a triangle are concurrent. For if M and N are the midpoints of the sides AB and AC respectively of a triangle ABC, then the triangles ABC and AMN are similar (SAS), and so MN and BC are parallel. The same holds for the other two sides. So ABC's medial triangle is homothetic to ABC itself.



The medians BN and CN, which connect corresponding pairs of points of the two triangles, are not parallel, for they meet inside ABC. It follows that we have collinearity rather than parallelism. So there is a centre of G of homothety of the two triangles, and all three medians must past through it. □

Example 6 (Existence of Euler line)**.** The centroid G, orthocentre H and circumcentre O of a triangle ABC are collinear. For O is the orthocentre of ABC's medial triangle. Hence the line OH must pass through the centre of homothety of ABC with its medial triangle, which is G.



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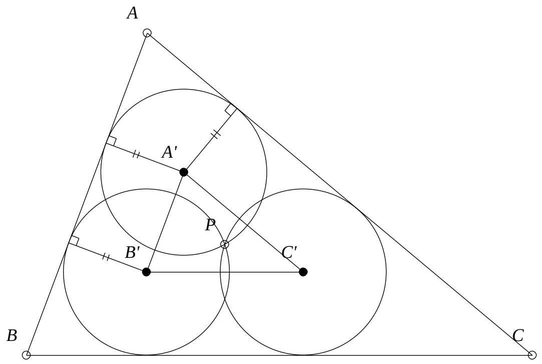
Exercise 4. A triangle ABC has incentre I, and its incircle is tangent to BC at X. Let S be the reflection of X in the perpendicular bisector of BC, and let T be the inversion of S through the point I. Show that A, S and T are collinear.

Example 7 (IMO 1981/1)**.** Three circles of equal radius and having some common point P are given inside triangle ABC, such that each side of ABC is tangent to two of the circles. Then P is collinear with the incentre I and circumcentre O of ABC.

For each of the circles must be tangent to two of ABC's sides, and each pair of sides of ABC must be tangent to exactly one of the three circles. Let A', B', C' be the centres of the circles tangent to AB and AC, AB and BC, AC and BC respectively. We will show that $A'B'C'$ is homothetic to ABC with centre of homothety I, and that P and O correspond under this homothety.

Indeed, as A' and B' are centres of circles of equal radii both tangent to AB, they must be equidistant from AB, and so the line $A'B'$ is parallel to AB; similarly, $A'C'$ is parallel to AC, and $B'C'$ to BC. So $A'B'C'$ and ABC are homothetic.



Next, as A' is the centre of a circle tangent to AB and AC, it is equidistant from AB and AC. A' also lies inside the sector $\angle BAC$. So A' lies on the internal bisector AI of $\angle A$; that is, I lies on AA'. Similarly, BB' and CC', lines connecting other corresponding pairs of points, also pass through I. So I is centre of homothety of ABC and $A'B'C'$.

As PA', PB', PC' are all radii of one of the circles, they must have equal length; hence P is the circumcentre of $A'B'C'$. It therefore corresponds under the homothety to O, the circumcentre of ABC, and hence P, I and O are collinear. □

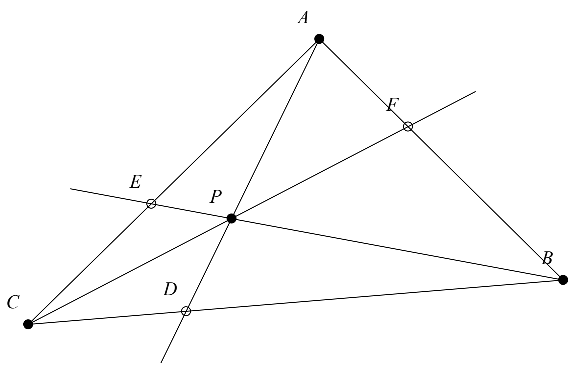
4 Ceva and Menelaus

In this section we use *directed line segments*. Choose a *positive direction* arbitrarily for each set of parallel lines. Then give a segment PQ positive *signed length* if it is positively directed, and negative signed length otherwise.

Theorem 2 (Ceva)**.** *Let ABC be a triangle. Let D, E, F be points on BC, CA, AB respectively. Suppose that D, E, F are distinct from A, B, C.*

Then AD, BE, CF are concurrent or parallel, if and only if, taking signed lengths,

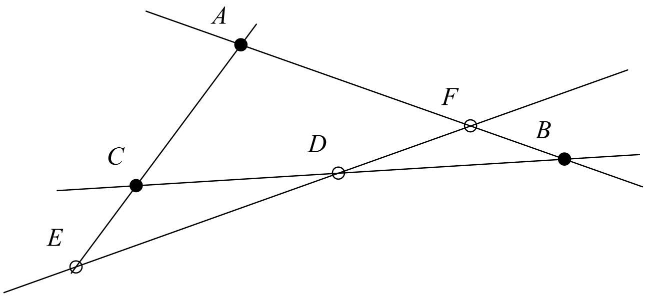
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$



Theorem 3 (Menelaus)**.** *Let ABC be a triangle. Let D, E, F be points on BC, CA, AB respectively. Suppose that D, E, F are distinct from A, B, C.*

Then D, E, F are collinear, if and only if, taking signed lengths,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$



Exercise 5 (Existence of Gergonne point)**.** Let X, Y and Z be the points at which the incircle of a triangle ABC touches its sides BC, CA and AB respectively. Show that AX, BY and CZ are concurrent.

Exercise 6 (Existence of Nagel point)**.** Let X, Y and Z be the points at which the A-, B- and C-excircles of a triangle ABC touch its sides BC, CA and AB respectively. Show that AX, BY and CZ are concurrent.

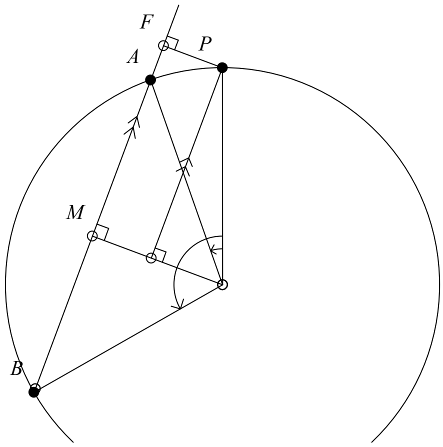
Example 8 (Existence of Simson line)**.** Let P be a point on the circumcircle of a triangle ABC. Let D, E and F be the feet of the three perpendiculars from P to BC, CA and AB respectively. Then D, E and F are collinear.

For this follows from Menelaus' theorem, where to make calculations easier we will assume without loss of generality that P, A, B, C are arranged around the circle counterclockwise in that order, and define α, β, γ respectively to be the measurements of the counterclockwise-directed angles $\angle POA$, $\angle POA$, $\angle POA$ (where O is the centre of the circle).

Indeed, let M be the midpoint of AB. Taking AB as positively-directed, we have

$$FM = R\sin\left(\frac{\alpha+\beta}{2}\right) \quad \text{and} \quad AB = 2R\sin\left(\frac{\beta-\alpha}{2}\right),$$

where R is the circumradius of ABC.



So

$$AF = AM - FM = R\left[\sin\left(\frac{\beta-\alpha}{2}\right) - \sin\left(\frac{\alpha+\beta}{2}\right)\right] = -2R\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right),$$

$$FB = MB + FM = R\left[\sin\left(\frac{\beta-\alpha}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right)\right] = 2R\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right).$$

Hence,

$$\frac{AF}{FB} = -\frac{\tan(\alpha/2)}{\tan(\beta/2)}.$$

Likewise,

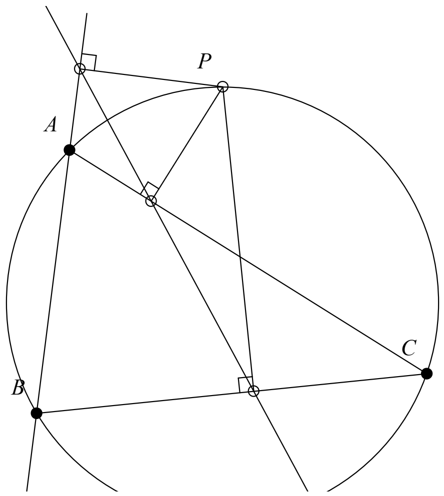
$$\frac{BD}{DC} = -\frac{\tan(\beta/2)}{\tan(\gamma/2)},$$

$$\frac{CE}{EA} = \left(\frac{AE}{EC}\right)^{-1} = -\frac{\tan(\gamma/2)}{\tan(\alpha/2)}.$$

Multiplying the three ratios together to apply the Menelaus test, we get

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1$$

as required. So D, E and F are collinear.



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5 Reflections

The reflection of a point P in a line l is the point P' on the perpendicular from P to l distinct from P, such that P and P' are equidistant from the foot of that perpendicular. (If P lies on l, then the reflection of P is just P.)

A surprisingly useful observation is that a line and its reflection, if not both parallel to the axis of reflection, must meet on the axis of reflection.

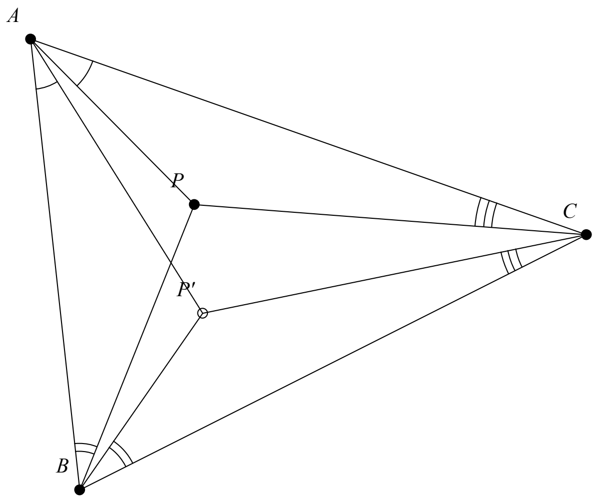
Exercise 7 (IMO 2004/5, easy direction)**.** Let $ABCD$ be a convex quadrilateral, in which the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$, equidistant from A and C, and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is cyclic.

6 Isogonal conjugates

Theorem 4 (Existence of isogonal conjugate)**.** *Let P be a point inside a triangle ABC. Let l_A be the reflection of the line AP through the internal angle bisector of $\angle A$. Define l_B and l_C similarly. Then l_A, l_B and l_C are concurrent; the point of concurrency is called the* isogonal conjugate *of P in ABC.*



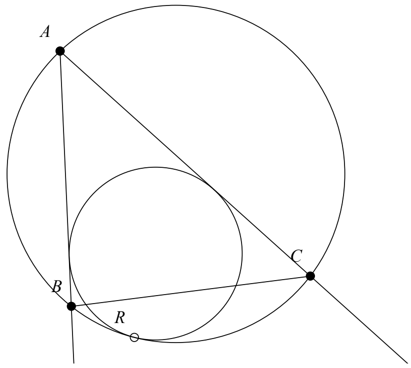
Example 9. The isogonal conjugate of the circumcentre is the orthocentre. The isogonal conjugate of the incentre is itself.

Exercise 8. Let ABC be an acute-angled triangle. Let S_A be the circle with diameter AR, where R is the foot of the altitude from A. Let S_A meet sides AB and AC at M and N respectively.

(a) Show that the triangles ABC and ANM are similar.

(b) Let L_A be the perpendicular from A to NM. Construct L_B and L_C analogously. Deduce that L_A, L_B and L_C are concurrent.

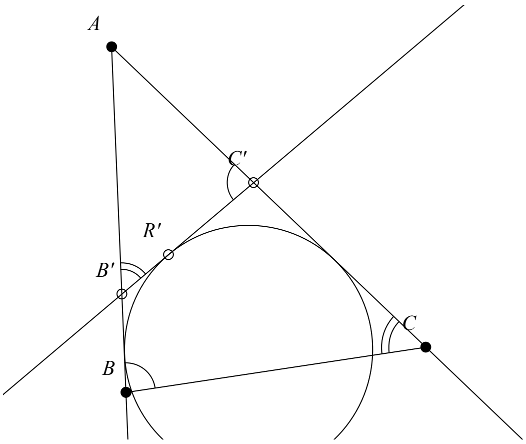
Example 10. Let Ω be the circumcircle of a triangle ABC. Let R be the point of tangency of Ω_A with Ω, where Ω_A is the (unique) circle tangent to AB and AC and internally tangent to Ω.



Points S and T are defined similarly with respect to vertices B and C. We will show that AR, BS and CT are concurrent.

In particular, recall from Exercise 6 that if X, Y and Z are the points of tangency of the A-, B- and C-excircles of ABC respectively with the sides AB, BC and CA, then AX, BY and CZ are concurrent; this point is called ABC's *Nagel point*. We will show that each of the lines AR, BS, CT passes through the isogonal conjugate of ABC's Nagel point.

Indeed, pick some inversion with centre A. Let B', C', R' be the images of B, C, R under this inversion; we have that B' lies on the line AB, and so on. The inversion sends the circle Ω to the line $B'C'$, reverses the orders of points on the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and preserves tangencies. Hence the image of Ω_A under the inversion is the A-excircle of triangle $AB'C'$, and the point R' is the point of tangency of this excircle with the side $B'C'$.



Finally, note that the triangles ABC and $AC'B'$ are similar, but oppositely oriented. The two A-excentre-tangency-points X and R' correspond under this similarity, so lines AX and AR' are each others' reflections in the internal angle bisector of $\angle A$. Since A, R and R' are collinear, we conclude that AR is AX's reflection in $\angle A$'s bisector. Likewise for BS and CT. So AR, BS and CT are concurrent at the isogonal conjugate of the point of concurrency of AX, BY and CT. □

7 Problems

1. (Swiss MO 2005, final round) Let ABC be an acute-angled triangle. Let M and N be arbitrary points on the segments AB and AC respectively, such that the circles with diameters BN and CM intersect at distinct points P and Q. Prove that the orthocentre of ABC lies on PQ.

2. (All-Russian Olympiad 2006) The angle bisectors of the angles ABC and BCA of a triangle ABC intersect the sides CA and AB at the points B_1 and C_1 and intersect each other at the point I. The line B_1C_1 intersects the circumcircle of triangle ABC at the points M and N. Prove that the circumradius of triangle MIN is twice the circumradius of triangle ABC.

3. (Balkan MO 1990) Let ABC be a triangle. Let DEF be a second triangle, whose vertices are the feet of the altitudes of ABC. Let GHI be a third triangle, whose vertices are the points of tangency of the incircle of DEF to its sides. Prove that the Euler lines of ABC and GHI coincide.

4. (IMO 1982/2) A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 opposite A_1, A_2, A_3 respectively. Let M_1 be the midpoint of a_1, and T_1 the point where the incircle of $A_1A_2A_3$ touches a_1. Then let S_1 be the reflection of T_1 in the interior bisector of A_1. The points M_2, M_3, T_2, T_3, S_2, S_3 are defined analogously. Prove that the lines M_1S_1, M_2S_2, M_3S_3 are concurrent.

5. (IMO Shortlist 2001) Let A' be the centre of the square inscribed in the acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B' and C' are defined similarly for inscribed squares with two vertices on the sides AC and AB respectively. Prove that the lines AA', BB', CC' are concurrent.

6. (IMO 2005/1) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.

   (a) Show that there exists a point P inside ABC, such that $A_1P = A_2P = B_1P = C_2P$.

   (b) Deduce that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

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