The Seven Colour Theorem

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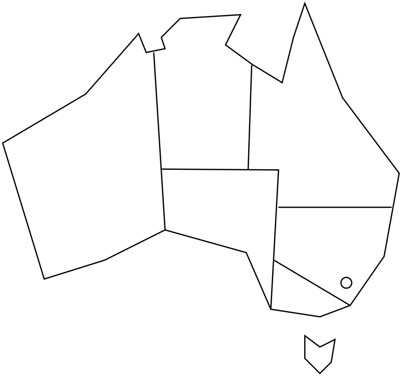
NZMOC Camp 2009

Outline

Map colouring

How many crayons do you need to colour Australia…



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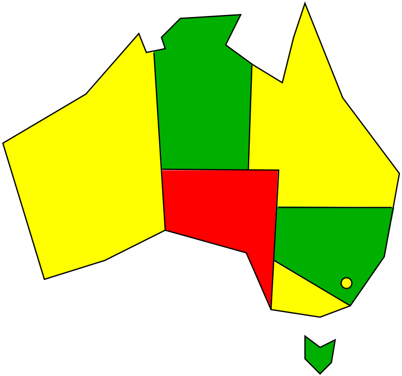

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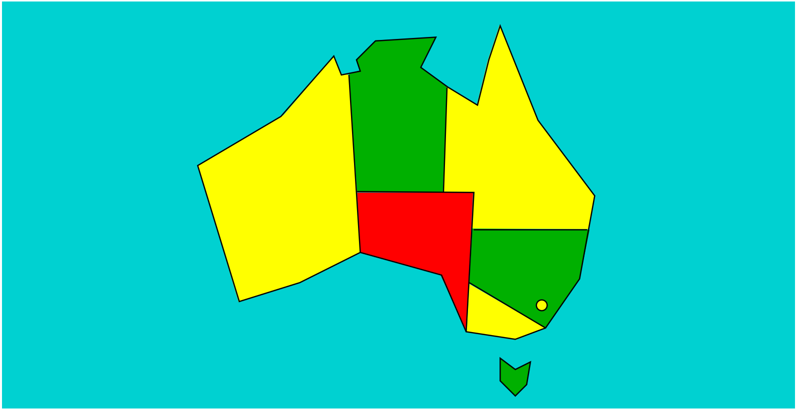

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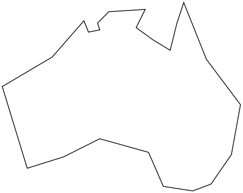
“Four colors suffice”

Theorem (Appel and Haken, 1976)

Four colours are necessary and sufficient to properly colour maps drawn in the plane.

- Some maps require four colours (easy!)



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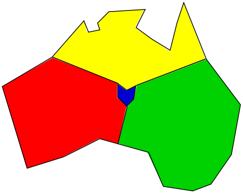

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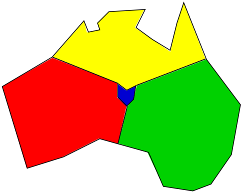

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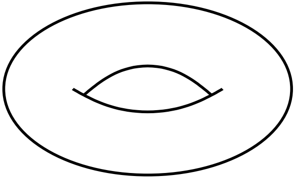

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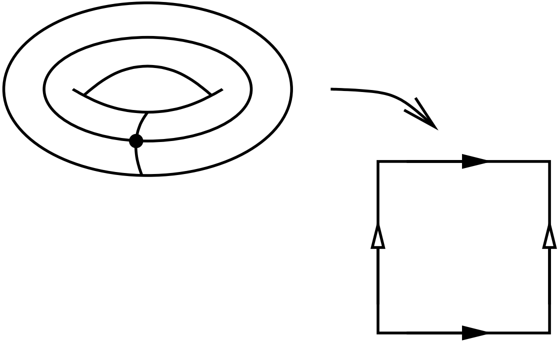

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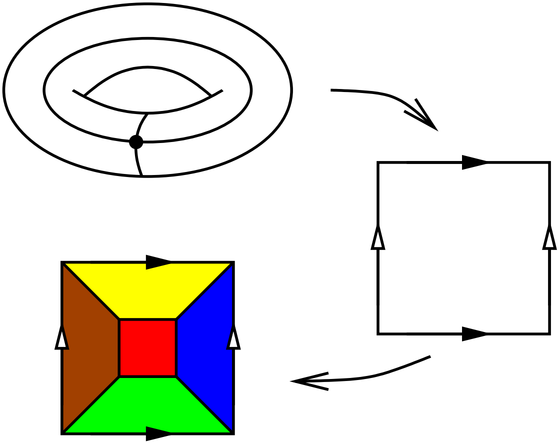


How many colours do we need??

The Seven Colour Theorem

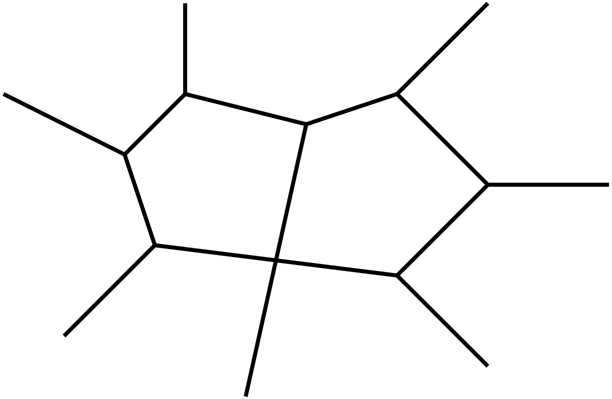
Theorem

Seven colours are necessary and sufficient to properly colour maps on a torus.

Steps:

1. Simplify!
2. Use the *Euler characteristic* to find the *average degree*.
3. Look at a minimal counterexample.
4. Prove necessity.

From maps to graphs



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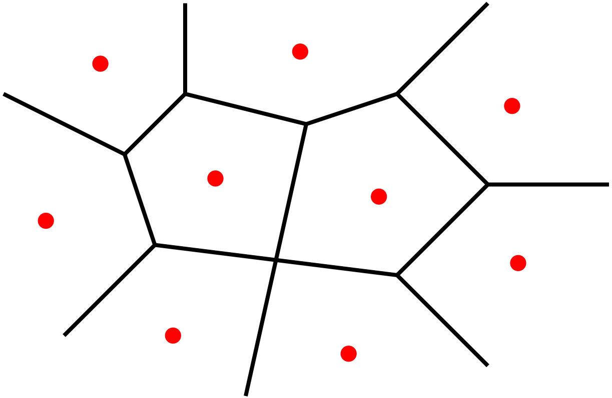

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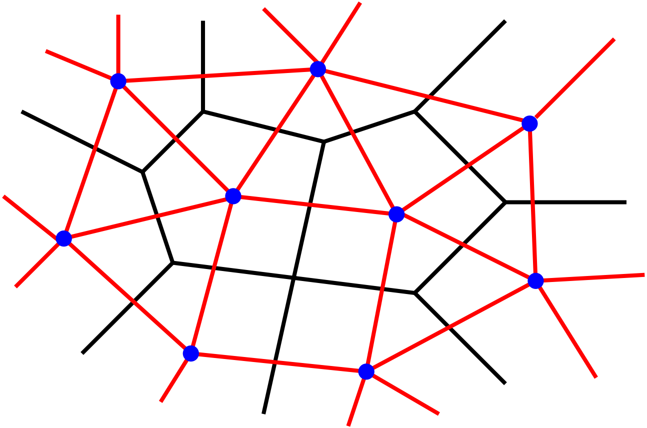

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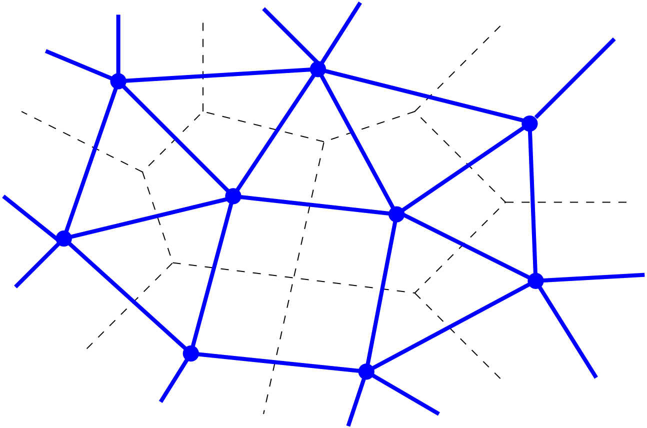


The *dual* of the map

Euler characteristic

- S a surface
- G a graph drawn on S so that
  - no edges or vertices cross or overlap
  - all regions (*faces*) are discs
  - there are

    V vertices, E edges, F faces.

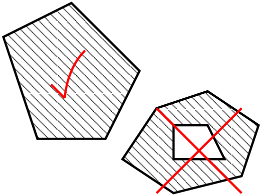
Definition

The *Euler characteristic* of S is $\chi(S) = V - E + F$.

Theorem

$\chi(S)$ depends only on S and not on G.

Euler characteristic



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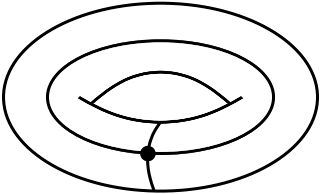
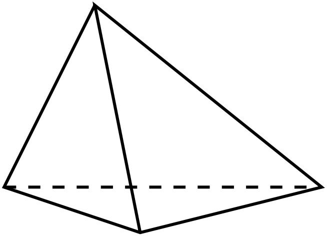


$\chi(\text{torus}) = 1 - 2 + 1 = 0$

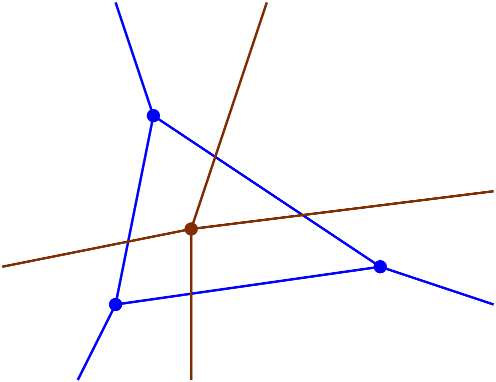


$\chi(\text{sphere}) = 4 - 6 + 4 = 2$

Proof of invariance

Given graphs G_1 and G_2, find a common refinement H.

- Subdivide edges
- Add vertices in faces
- Subdivide faces.



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ΔV	ΔE	ΔF	$\Delta \chi$
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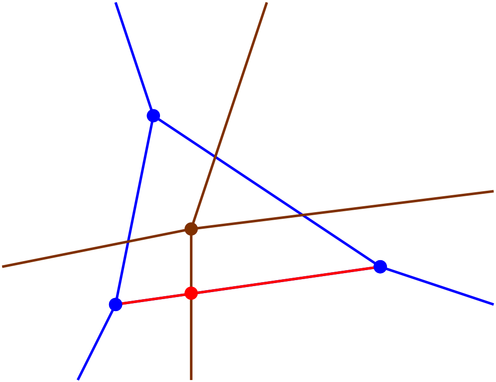

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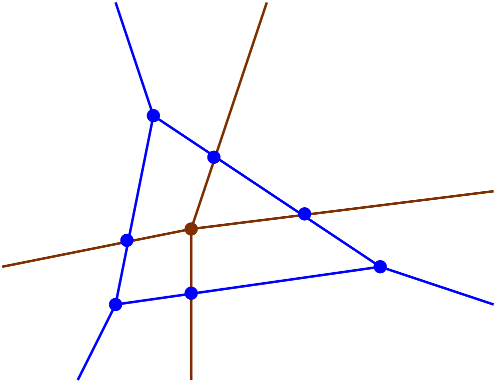

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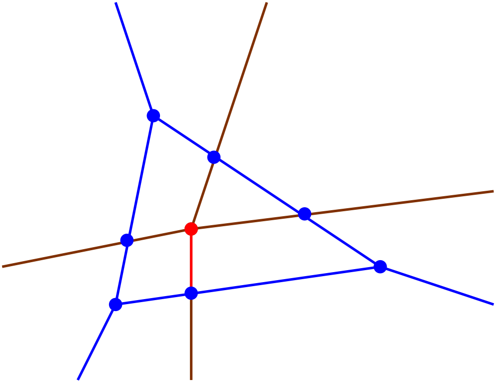

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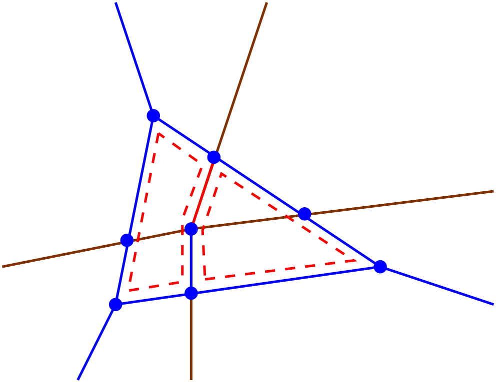

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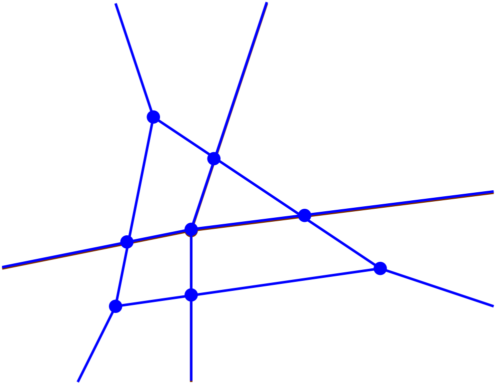
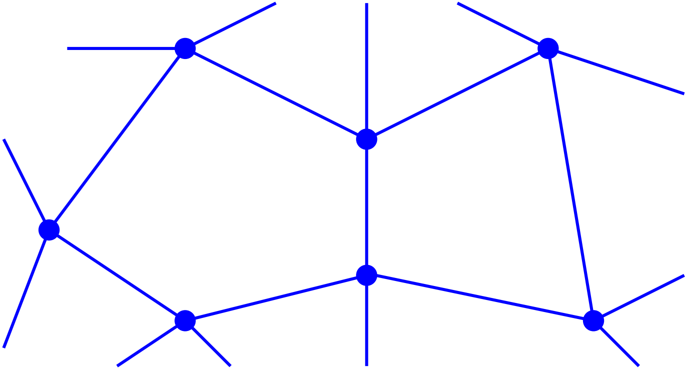


Don’t wait—triangulate!

We may assume all faces are triangles:



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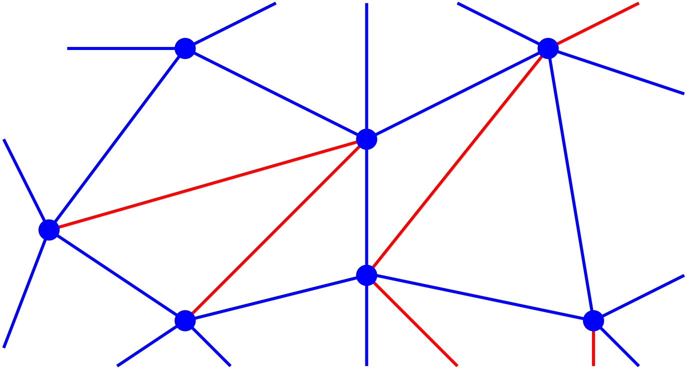
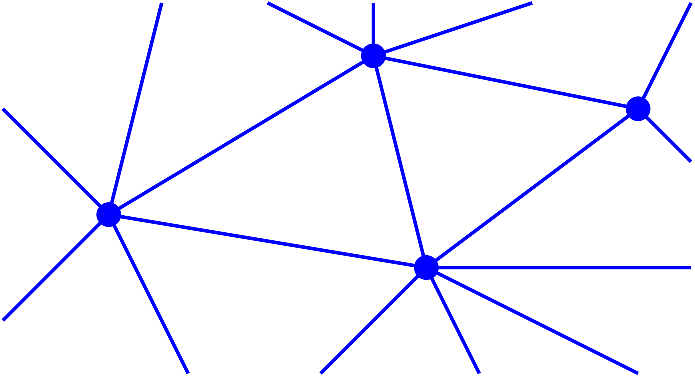


Count two ways twice

When all faces are triangles:

$$3F = 2E = \sum_{v} \text{degree}(v)$$



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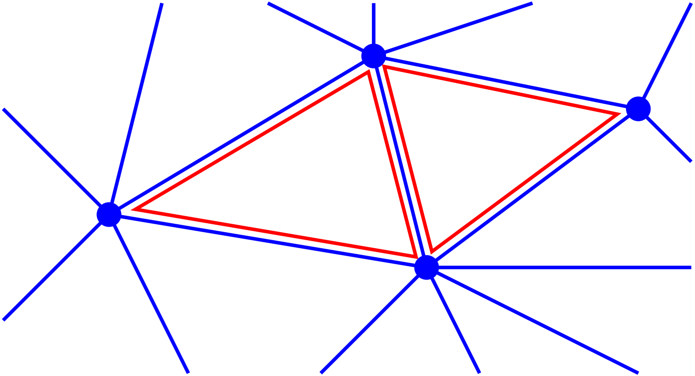

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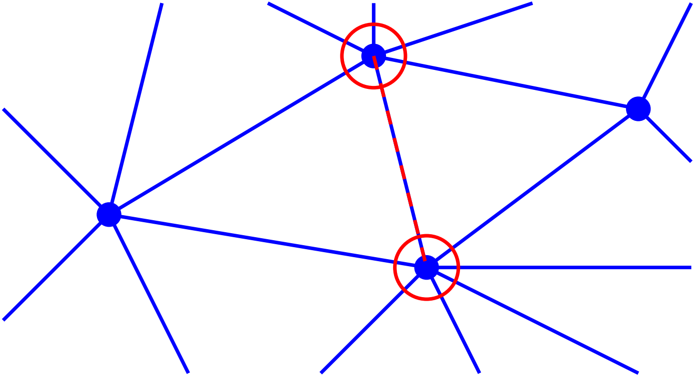

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$V - E + F = 0$ and $3F = 2E = \sum_v \text{degree}(v)$ give

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$$\implies \quad \frac{1}{V}\sum_v \text{degree}(v) = 6$$

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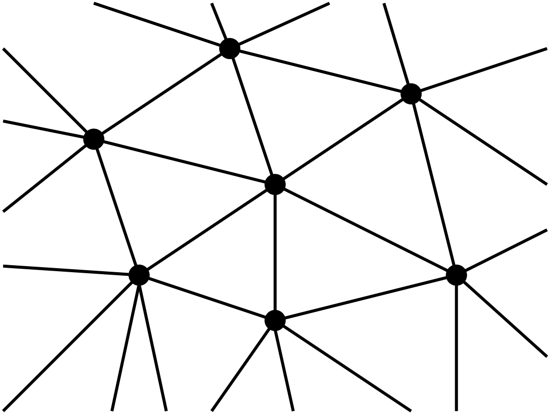

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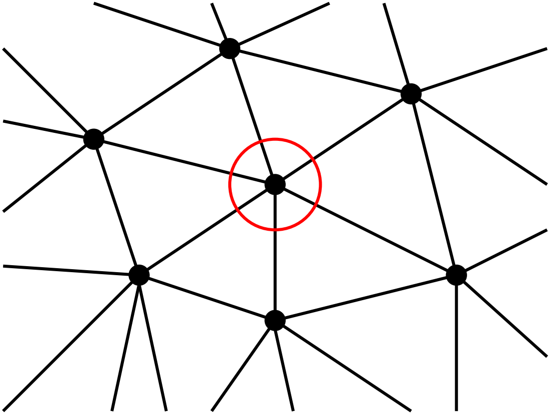

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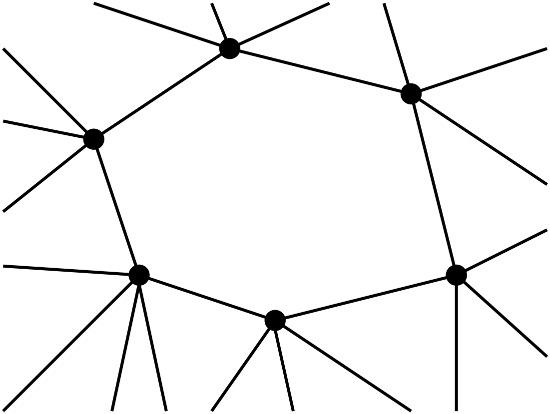

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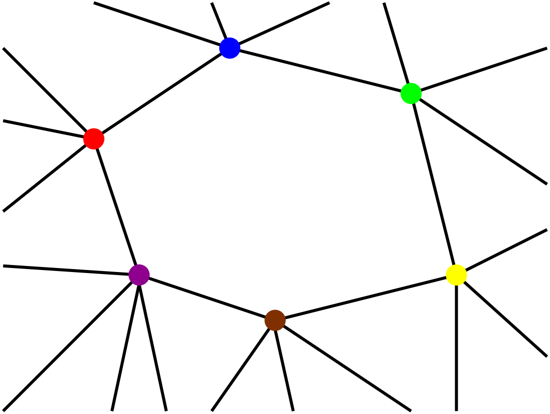

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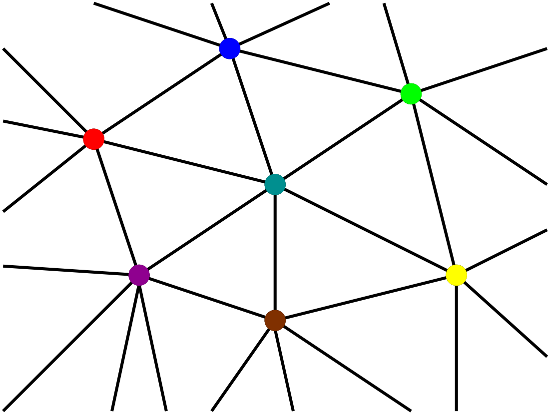
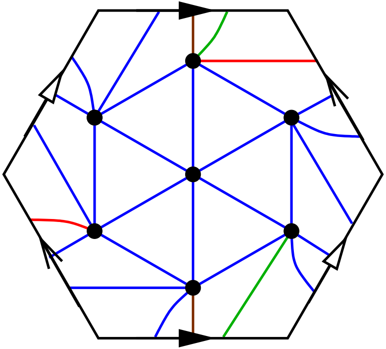


…why, it's not a counterexample at all!

Seven are necessary



The complete graph K_7 embedded on the torus.

The Four and Five Colour Theorems

Five colours:

- A triangulation of the plane has a vertex v of degree at most five.
- “Kempe chains” reduce the number of colours needed for v’s neighbours to four.

Four:

- Find an *unavoidable* set of configurations, and show that none can occur in a minimal counterexample.
- The proof has been simplified by Robinson, Sanders, Seymour and Thomas (1996), but still requires a computer.
- Robinson et. al. use 633 configurations in place of Appel and Haken’s 1476.

The Heawood bound

Theorem (Heawood, 1890, via average degree arguments))

Maps on a surface of Euler characteristic $\chi \leq 1$ *require at most*

$$\left\lfloor \frac{7+\sqrt{49-2\chi}}{2} \right\rfloor$$

colours.

- The Klein bottle has $\chi = 0$ but requires only six colours (Franklin, 1934)
- Bound is otherwise tight (Ringel and Youngs, 1968)