New Zealand Mathematical Olympiad Committee

A Test for Orthogonality

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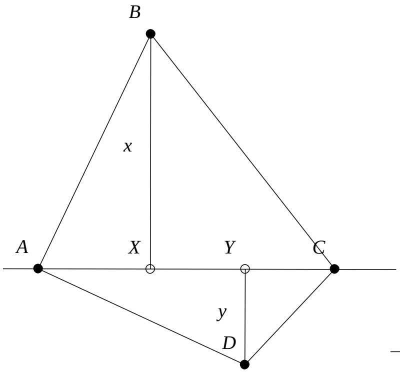
1 Introduction

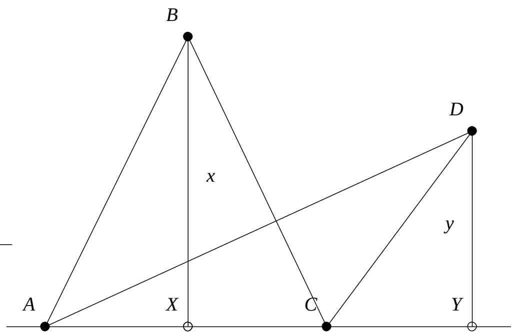
These notes outline a useful theorem in geometry. It provides a method for proving that two lines intersect at right angles without explicitly calculating the lines' point of intersection.

Theorem 1. *Two coplanar line segments AC and BD are perpendicular if and only if $AB^2+CD^2 = AD^2+CB^2$.*

2 A proof using lengths

Proof. Let X and Y be the feet of the perpendiculars from B and D respectively to AC, and let x and y be the lengths of these perpendiculars. By Pythagoras, $AB^2 = (x^2 + AX^2)$, etc., so $AB^2 + CD^2 = AD^2 + CB^2$ if and only if $AX^2 + CY^2 = AY^2 + CX^2$.





If $AX^2 + CY^2 = AY^2 + CX^2$, then

$$(AX - AY)(AX + AY) = (CX - CY)(CX + CY).$$

Let $XY = d$, $AX = a$, $CX = c$. There are three possibilities for the positioning of A:

- A lies past X on the ray YX, and $(AX - AY)(AX + AY) = -d(d + 2a)$.
- A lies between X and Y, and $(AX - AY)(AX + AY) = (2a - d)d$.
- A lies past Y on the ray XY, and $(AX - AY)(AX + AY) = d(2a + d)$.

The equivalent is true of C, and so, swapping A and C if necessary, we have six cases:

1. A far side of X, C far side of X. $-d(d + 2a) = -d(d + 2c)$.
2. A far side of X, C between X and Y. $-d(d + 2a) = (2c - d)d$.
3. A far side of X, C far side of Y. $-d(d + 2a) = d(2c + d)$.
4. A between X and Y, C between X and Y. $(2a - d)d = (2c - d)d$.

5. A between X and Y, C far side of Y. $(2a-d)d = d(2c+d)$.
6. A far side of Y, C far side of Y. $d(2a+d) = d(2c+d)$.

Suppose $AX^2 + CY^2 = AY^2 + CX^2$ but AC and BD are not perpendicular. Then $d > 0$, so we can divide through by d in whichever of the six equations is valid. It immediately follows that cases 1, 4, 6 are degenerate since they imply $AX = CX$ and A and C on the same side of X, and hence that A and C coincide.

Case 3 implies $-(d+2a) = 2c+d$, a contradiction since $-(d+2a) \leq -d < 0 < d \leq 2c+d$. Case 2 implies $a = -c$, a contradiction unless we have the degenerate case of A and C coinciding at X. Case 5 implies that $d = -d$ and hence $d = 0$, contrary to assumption.

It follows that if $AX^2 + CY^2 = AY^2 + CX^2$ then AC and BD are perpendicular. Conversely, if AC and BD are perpendicular, then X and Y coincide, so $AX^2 + CY^2 = AY^2 + CX^2$. □

(This pure-geometry proof can be shortened somewhat by treating AC, AX etc. as directed rather than undirected line segments.)

3 A proof using vectors

Proof. Choose some arbitrary origin O. AC and BD are perpendicular if and only if $(\overrightarrow{OA}-\overrightarrow{OC})\cdot(\overrightarrow{OB}-\overrightarrow{OD}) = 0$; that is,

$$\overrightarrow{OA}\cdot\overrightarrow{OB} + \overrightarrow{OC}\cdot\overrightarrow{OD} = \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OC}\cdot\overrightarrow{OB}.$$

$AB^2 + CD^2 = AD^2 + CB^2$ if and only if

$$||\overrightarrow{OA}-\overrightarrow{OB}||^2 + ||\overrightarrow{OC}-\overrightarrow{OD}||^2 = ||\overrightarrow{OA}-\overrightarrow{OD}||^2 + ||\overrightarrow{OC}-\overrightarrow{OB}||^2;$$

that is,

$$(\overrightarrow{OA}-\overrightarrow{OB})\cdot(\overrightarrow{OA}-\overrightarrow{OB}) + (\overrightarrow{OC}-\overrightarrow{OD})\cdot(\overrightarrow{OC}-\overrightarrow{OD}) = (\overrightarrow{OA}-\overrightarrow{OD})\cdot(\overrightarrow{OA}-\overrightarrow{OD}) + (\overrightarrow{OC}-\overrightarrow{OB})\cdot(\overrightarrow{OC}-\overrightarrow{OB}),$$

or

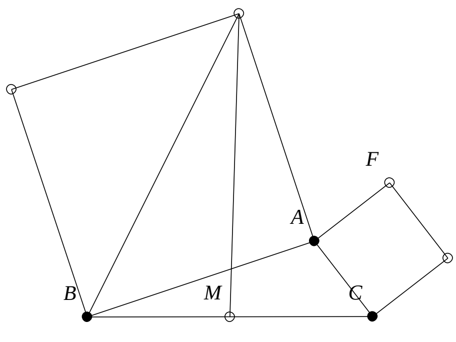
$$-2\overrightarrow{OA}\cdot\overrightarrow{OB} - 2\overrightarrow{OC}\cdot\overrightarrow{OD} = -2\overrightarrow{OA}\cdot\overrightarrow{OD} - 2\overrightarrow{OC}\cdot\overrightarrow{OB}.$$

These are clearly equivalent. □

4 Examples

Example 1. Let ABC be any triangle. Two squares $BAEP$ and $CAFR$ are constructed externally to ABC. Let M be the midpoint of BC. Show that AM is perpendicular to EF.

Solution: We want to prove that $ME^2 + AF^2 = MF^2 + AE^2$.



The Cosine Rule gives

$$
\begin{aligned}
ME^2 &= MB^2 + BE^2 - 2MB \cdot BE \cos \angle CBE \\
&= \frac{1}{4}a^2 + 2c^2 - 2 \cdot \frac{1}{2}a \cdot \sqrt{2}c \cdot \frac{1}{\sqrt{2}}(\cos \angle CBA - \sin \angle CBA).
\end{aligned}
$$

Hence

$$
\begin{aligned}
ME^2 + AF^2 &= \frac{1}{4}a^2 + 2c^2 - ac\cos \angle CBA + ac \sin \angle CBA + b^2 \\
&= -\frac{1}{4}a^2 + \frac{3}{2}c^2 + \frac{1}{2}(a^2 + c^2 - 2ac\cos \angle CBA) + 2(\frac{1}{2}ac \sin \angle CBA) + b^2 \\
&= -\frac{1}{4}a^2 + \frac{3}{2}c^2 + \frac{3}{2}b^2 + 2 \triangle ABC.
\end{aligned}
$$

By symmetry, $MF^2 + AE^2$ is equal to the same value, so the orthogonality holds. □

Example 2. Let ω_1 and ω_2 be two circles with centres O_1 and O_2 respectively. Show that the set of all points which have equal powers with respect to ω_1 and ω_2 (known as the *radical axis* of ω_1 and ω_2) is a straight line perpendicular to O_1O_2.

Solution: It suffices to prove that for any two such points P and Q, PQ is perpendicular to O_1O_2.

For this, let r_1 and r_2 be the radii of ω_1 and ω_2 respectively. Then the powers of P and Q with respect to ω_1 and ω_2 are $O_1P^2 - r_1^2$, $O_2P^2 - r_2^2$, $O_1Q^2 - r_1^2$, $O_2Q^2 - r_2^2$.

$$
\begin{aligned}
O_1P^2 + O_2Q^2 &= (O_1P^2 - r_1^2) + (O_2Q^2 - r_2^2) + r_1^2 + r_2^2 \\
&= (O_2P^2 - r_2^2) + (O_2Q^2 - r_2^2) + r_1^2 + r_2^2 \\
&= O_2P^2 + O_1Q^2.
\end{aligned}
$$

It follows that PQ and O_1O_2 are perpendicular. □

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