



New Zealand Maths Olympiad Committee  
2009 Maths Gymnastics  
Christchurch, Thursday 14 January

1. Simplify

$$\sin^2 70^\circ \times \sin^2 50^\circ \times \sin^2 10^\circ$$

2. Simplify

$$\sin \frac{3\pi}{10} - \sin \frac{\pi}{10}$$

3. Suppose  $a$ ,  $b$  and  $c$  are the sides of a triangle such that

$$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

Find one of the angles of this triangle.

4. Prove that

$$\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{99}{100} < \frac{1}{10}$$

5. Find all pairs of real numbers  $(x, y)$  such that the following two conditions both hold.

$$\begin{aligned}x^4 + y^4 &= 17 \\x + y &= 3\end{aligned}$$

6. Solve the equation

$$26 \sin^2 x^2 + 12 \cos 2x + 5 \sin 2x = 13$$

7.  $ABCD$  is a trapezium with  $BC \parallel AD$ ,  $AB = 9$  and  $CD = 5$ . The bisector of  $\angle D$  intersects the bisectors of  $\angle A$  and  $\angle C$  at points  $M$  and  $N$  respectively. The bisector of  $\angle B$  intersects the bisectors of  $\angle A$  and  $\angle C$  at points  $L$  and  $K$  respectively. Suppose that  $K$  lies on  $AD$ . Let  $P$  be the intersection of lines  $LN$  and  $AB$ , and let  $Q$  be the intersection of lines  $KM$  and  $BC$ .

- Find the ratios  $AP : PB$  and  $BQ : QC$ .
- If  $LM : KN = 3 : 7$ , find the ratio  $MN : KL$ .

8. Find one of the roots of the following polynomial

$$x^3 + 2\sqrt{3}x^2 + 3x + \sqrt{3} - 1 = 0$$

9. Solve the equation:

$$x^2 + \frac{25x^2}{(5+2x)^2} = \frac{74}{49}$$

10. Prove the inequality

$$\cos(x) + x \sin(x) > 1$$

where  $0 < x \leq \frac{\pi}{2}$

11. Suppose  $\alpha$  and  $\beta$  are two angles of a given triangle and

$$\cos \alpha + \cos \beta - \cos(\alpha + \beta) = \frac{3}{2}$$

Find the angles of the triangle.