New Zealand Maths Olympiad Committee
September Problems 2008
Due: 25 October

Junior Division

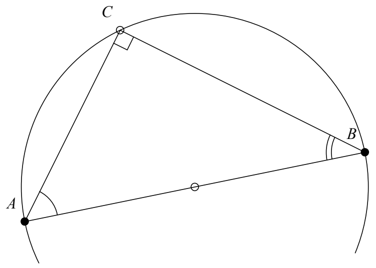
1. *When amoebas of a particular exotic species reproduce, the parent amoeba splits apart into four identical baby amoebas. Over time, the members of a colony of these amoebas occasionally reproduce, and the colony's size grows. Suppose the colony contains four amoebas to start with. Can its number of members ever reach exactly 2009?*

   Solution: Notice that whenever an amoeba reproduces, it disappears but four more new amoebas appear. So at every reproduction the number of amoebas increases by three.

   However, the colony started with four amoebas, and $2009 - 4 = 2005$ isn't a multiple of three. So the colony will never reach exactly 2009 amoebas.

2. *An isosceles triangle is inscribed in a circle, so that one of the sides of the triangle is the circle's diameter. What are the angles of the triangle?*

   Solution: Let AB be the side of the triangle that is a diameter, and let C be the triangle's third vertex. An angle inscribed in a semicircle is 90^o. So $\angle ACB$ is 90^o.



   An isosceles triangle is one with two of its angles equal. In this case, that means that either

   - one of the two angles $\angle ABC$ is $\angle BAC$ is equal to $\angle ACB = 90^o$; or,
   - the two angles $\angle ABC$ and $\angle BAC$ are equal.

   The first case is impossible: if two angles in a triangle are both 90^o, then the third angle needs to be $180^o - 90^o - 90^o = 0^o$ and the triangle is degenerate.

   So the second case must happen. This means that the other two angles $\angle ABC$ and $\angle BAC$ of the triangle are both $(180^o - 90^o)/2 = 45^o$.

3. *Show that*

$$n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \geq (n+1)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

   for all natural numbers $n \geq 1$.

   Solution: Write X for $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. Now, each of $\frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}$ is less than 1, so

$$n-1 \geq \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

and hence

$$n \geq \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)+\frac{n+1}{n+1}.$$

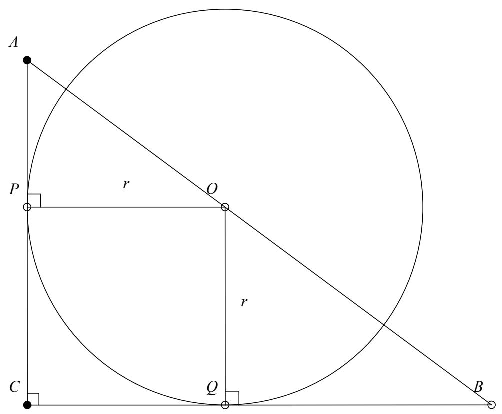
Adding nX to each side gives

$$n+nX \geq X+nX+\frac{n+1}{n+1}$$

and, factorizing, we get $n(1+X) \geq (n+1)(X+\frac{1}{n+1})$ which is what is needed.

4. *A right-angled triangle has legs of length a and b. A circle of radius r touches the two legs and has its centre on the hypotenuse. Show that $\frac{1}{a}+\frac{1}{b}=\frac{1}{r}$.*

Solution: Let O be the centre of the circle, let the vertices opposite the sides with lengths a and b be A and B respectively, and let the third vertex be C. Let P and Q be where the circle touches sides AC and BC respectively, so $OP = OQ = r$; notice that because tangents meet a circle at right angles, $\angle APO$ and $\angle OQB$ are right angles. So $OPCQ$ is a square. $AP = AC - PC = b - r$, and $BQ = BC - QC = a - r$.



The areas of triangles APO, OQB and ACB are $\frac{1}{2}r(b-r)$, $\frac{1}{2}r(a-r)$ and $\frac{1}{2}ab$ respectively, and the area of the square $OPCQ$ is r^2. As the triangles APO and OQB together with the square $OPCQ$ make up the triangle ACB,

$$\begin{aligned}\frac{1}{2}ab &= \frac{1}{2}r(b-r)+\frac{1}{2}r(a-r)+r^2\\ &= \frac{1}{2}r(a+b)\end{aligned}$$

Dividing through by $\frac{1}{2}abr$,

$$\frac{1}{r}=\frac{1}{a}+\frac{1}{b}.$$

Alternatively (keeping the same notation as before), notice that because of all the right angles, APO and OQB are both similar to ACB, and so are similar to each other. Comparing the ratios of corresponding sides,

$$\begin{aligned} \frac{AP}{PO} &= \frac{OQ}{QB} \\ \frac{b-r}{r} &= \frac{r}{a-r} \\ (b-r)(a-r) &= r^2 \\ ab &= (a+b)r \end{aligned}$$

Dividing through by abr,

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}.$$

5. *Michael plans to buy a set of weights for his balance scale, so that the following conditions are satisfied:*

   (a) The set contains weights of at least five different masses.

   (b) For any two weights in the set, two more weights from the set can be picked, so that the sums of the masses of the two pairs of weights are the same.

   He moreover wants to buy as few weights as possible. At least how many weights must this be?

   Solution: Let $a_1 < a_2 < \cdots < a_k$ be the masses of weight in the set Michael buys. Then for any pair a_i, a_j from the set with $i > 2$, its sum $a_i + a_j$ is greater than $a_1 + a_2$. Moreover $a_2 + a_2 > a_1 + a_2$, and $a_1 + a_1 < a_1 + a_2$. So the only pairs of weights from the set with total mass $a_1 + a_2$ are the pairs that contain one weight of mass a_1 and one mass of weight a_2. The set must therefore include at least two masses of weight a_1 and least two masses of weight a_2.

   Now, for any pair a_i, a_j from the set with $i > 1$, its sum $a_i + a_j$ is greater than $a_1 + a_1$. So the only pairs of weights from the set with total mass $a_1 + a_1$ are those consisting of two weights of mass a_1. The set must therefore include at least four masses of weight a_1.

   The same logic shows that the set must contain at least four masses of weight a_k and at least two of weight a_{k-1}. The set contains weights of at least five different masses, so $k - 1 > 3$. So the set must contain at least $4 + 2 + 1 + 2 + 4 = 13$ weights.

   Conversely, the set of thirteen weights $1, 1, 1, 1, 2, 2, 3, 4, 4, 5, 5, 5, 5$ will for instance do.

6. Integer-dividing *an integer P by a natural number Q yields two other integers: the* remainder *of P on division by Q, which is at least* 0 *and at most* $Q - 1$, *and the* quotient, *which is the greatest integer no bigger than the rational number P/Q. For instance, dividing* 17 *by* 6 *gives quotient* 2 *and remainder* 5.
   Let n be any natural number. For each factor d of $n + 1$, integer-divide n by d, and record the quotient and remainder. Prove that the sets of quotients and remainders you obtain by this process are the same.

   Solution: Let a be any divisor of $n + 1$, and let $b = (n + 1)/a$. It turns out that the quotient of n on integer-division by b is the same as the remainder of n on division by a, and vice versa.

   To show this, write $n = ab - 1 = a(b - 1) + (a - 1)$. Here $0 \leq a - 1 < a$, so $a - 1$ is the remainder of n on division by a, and $b - 1$ the quotient.

Senior Division

1. *For which prime numbers p and q (if any), is $5p^2q + 16pq^2$ a perfect square?*

   Solution: First suppose that $p = q$ were possible. Then $21p^3$ would have to be a perfect square, but to obtain an even number of factors of 3 would require $p = 3$, while to obtain an even number of factors of 7 would require $p = 7$. So, we may assume that $p \neq q$. Now:

   $$5p^2q + 16pq^2 = pq(5p + 16q).$$

   As this is supposed to be a square, and is clearly a multiple of p it must be a multiple of p^2. So, p is a divisor of $q(5p + 16q)$. But $p \neq q$, so p is a divisor of $5p + 16q$, hence of $16q$, and hence of 16. So, p must be 2. Similarly, q must be 5. Now we need to check that this works:

   $$2 \times 5 \times (5 \times 2 + 16 \times 5) = 2 \times 5 \times 90 = 30^2.$$

   So, there is exactly one possibility, $p = 2$, $q = 5$.

2. *Let $c_1, c_2, c_3, \ldots, c_{2009}$ be a sequence of real numbers such that $|c_n - c_{n+1}| \leq 1$ for $1 \leq n \leq 2008$. Show that:*

   $$\left| \frac{c_1 + c_2 + \cdots + c_{2009}}{2009} - \frac{c_1 + c_2 + \cdots + c_{2008}}{2008} \right| \leq \frac{1}{2}.$$

   Solution: We first observe that for $i \leq j$,

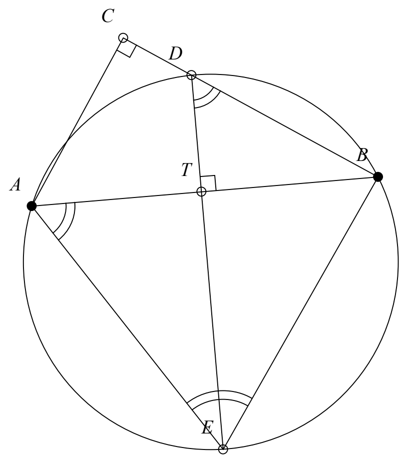
   $$\begin{aligned} |c_i - c_j| &= |c_i - c_{i+1} + c_{i+1} - c_{i+2} + \cdots + c_{j-1} - c_j| \\ &\leq |c_i - c_{i+1}| + |c_{i+1} - c_{i+2}| + \cdots + |c_{j-1} - c_j| \\ &\leq 1 + 1 + \cdots + 1 = j - i. \end{aligned}$$

   Now:

   $$\begin{aligned} & \left| \frac{c_1 + c_2 + \cdots + c_{2009}}{2009} - \frac{c_1 + c_2 + \cdots + c_{2008}}{2008} \right| \\ = & \frac{1}{2008 \times 2009} |2008c_{2009} - c_1 - c_2 - \cdots - c_{2008}| \\ \leq & \frac{1}{2008 \times 2009} (|c_{2009} - c_1| + |c_{2009} - c_2| + \cdots + |c_{2009} - c_{2008}|) \\ \leq & \frac{1}{2008 \times 2009} (2008 + 2007 + \cdots + 1) = \frac{1}{2}. \end{aligned}$$

3. *Let ABC be a right-angled triangle with right angle at C. Pick a point D on the segment BC. Let E be a point on the circumcircle ω of ABD, such that DE is perpendicular to AB. Prove that $\angle BAE = \angle BEA$ if and only if AC is tangent to ω.*

   Solution: Let T be the intersection of DE and AB. First suppose that $\angle BAE = \angle BEA = \alpha$. Then $\angle BDT = \angle BDE = \alpha$ since this is an angle on chord BE.



Now $\angle DBT = 90° - \alpha$ and so $\angle BAC = \alpha$. This makes $\angle BAC$ and $\angle BEA$ equal, and so AC must be tangent to ω.

Conversely, if AC is tangent to ω, we can reverse this angle-chase. Let $\alpha = \angle CAB$. Then $\angle CBA = 90° - \alpha$, whence $\angle BDE = \alpha$ and hence $\angle BAE = \alpha$. But tangency implies $\angle BEA = \angle BAC$, and so $\angle BEA = \angle BAE$.

4. *Let $n \geq 3$ be an odd integer. Determine the maximum possible value of the sum:*

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \cdots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|}$$

where $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$.

Solution: The maximum is $n - 2 + \sqrt{2}$ and this can be achieved if we set $x_1 = 1/2$ and then the remaining x's alternately 0 and 1.

Suppose that any sequence a_1, a_2, $\ldots$, a_n (all between 0 and 1) achieved a larger value. Among such sequences choose one in which the number of terms equal to 1 or to 0 is as large as possible.

First note that consecutive a's cannot be equal (otherwise the sum is at most $n-1$ which is too small.) So, in particular, not all the a's are 0 or 1. Suppose then that $0 < a_i < 1$. It cannot be the case (indexing cyclically) that $a_{i-1}, a_{i+1} < a_i$ or else we could change the sequence making $a_i = 1$, increasing the sum, and increasing the number of 1's or 0's. Similarly, we cannot have $a_{i-1}, a_{i+1} > a_i$. So, necessarily $a_{i-1} < a_i < a_{i+1}$ or vice versa. But then:

$$\frac{\sqrt{|a_{i-1} - a_i|} + \sqrt{|a_i - a_{i+1}|}}{2} \leq \sqrt{\frac{1}{2}|a_{i-1} - a_i + a_i - a_{i+1}|} \leq \frac{1}{\sqrt{2}},$$

(because $(\sqrt{x} + \sqrt{y})/2 \leq \sqrt{(x+y)/2}$.) So

$$\sqrt{|a_{i-1} - a_i|} + \sqrt{|a_i - a_{i+1}|} \leq \sqrt{2}.$$

The remaining $n-2$ terms contribute at most $n-2$, and so the total sum is at most $n-2+\sqrt{2}$.

5. *Determine the minimum possible value of the expression $|n^2 - 5^{4m+3}|$ for non-negative integers m and n.*

Solution: It is easy enough to make the value 4 (take $n = 11$ and $m = 0$) and we now show that it cannot be smaller than 4. First, it can't be 0 since 5^{4m+3} is not a perfect square. If

$n^2 > 5^{4m+3}$, then since n^2 always leaves a remainder of 0, 1 or 4 on division by 5, the only possible case where the difference is less than 4 would be: $n^2 - 5^{4m+3} = 1$. But then

$$5^{4m+3} = n^2 - 1 = (n-1)(n+1)$$

And it's certainly impossible for both $n-1$ and $n+1$ to be powers of 5.

Now supose that $5^{4m+3} > n^2$. Again, the only case we need to worry about is $5^{4m+3} - 1 = n^2$. But then:

$$n^2 = 5^{4m+3} - 1 = (5-1)(5^{4m+2} + 5^{4m+1} + \cdots + 1)$$

so the second factor is a perfect square. But, its remainder on division by 4 is 3 (it is the sum of $4m+3$ terms each leaving remainder 1), and perfect squares leave remainder 0 or 1 on division by 4. So, that's not possible either, and we're done.

6. *Michael's mother likes to keep him busy with an odd form of solitaire. To set up the game she places coins on some of the squares of a normal* 8×8 *chessboard. Michael plays by adding one coin at a time, always placing coins only on squares which already have at least two adjacent squares containing coins. (Two squares are adjacent if they share an edge, but not if they only share a vertex.)*

   Michael wins when he's placed a coin on every square of the board. What is the minimum number of coins that Michael's mother can place on the board to start with, so that it is still possible for Michael to win?

   Solution: The minimum number is 8. Many arrangements of 8 coins work, for example filling one of the diagonals (Michael can then fill above and below the diagonal, working outwards.)

   To see that 8 coins are required we consider the notion of a *split* edge. A split edge is one which on one side has a square containing a coin, but not on the other side. In particular edges on the boundary of the board can be split if the square they are part of contains a coin.

   Placing a coin always eliminates at least two split edges. However, it can only create at most two split edges (the two other edges of the square where a coin was just placed.) So, the number of split edges can never go up. Since the full board has 32 split edges (all the ones on the boundary of the board), the initial board must have at least 32 split edges. But, each square can have at most 4 split edges, so there must have been at least 8 occupied squares to begin with.