1. We know from intersecting secants that

$AE \times BE = DE \times CE$

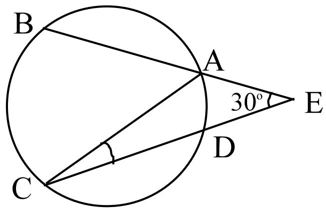
$AE \times (AE+AB) = DE \times (DE+CD)$

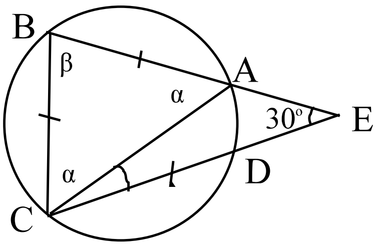
Since $CD = AB$ given

Then it follows that above equation can only be true if $AE=DE$.

This implies $BE = CE$

also we know $BC = AB$





Let $\triangle BCA = \triangle BAC = \alpha$ and $\triangle BAC = \beta$

since $\triangle BEC$ is isosceles, $\beta = 75^o$ (base angle)

and since $2\alpha + 75^o = 180$

then $\alpha = 52.5^o$

and $\triangle ACE = 75^o - 52.5^o$

$= 22.5^o$

[Heyang Li]

2. Given $a + b + c = 7 \Rightarrow a + ar + ar^2 = 7$

$ar^2 + ar + a - 7 = 0$

$$r = \frac{-a \pm \sqrt{28a - 3a^2}}{2a}$$

Since $a \in I$, then $r \in \mathbf{Q} \Rightarrow \triangle = 28a - 3a^2$ is a positive square number

and $28a - 3a^2 > 0 \Rightarrow 0 < a < 9$

the only values of $a \in I$ which give $\triangle$ a square number are 1,4, 7 or 9.

If $a = 1$, $\triangle = 25$, $r = 2$ or -3 giving sequence 1, 2, 4 and 1, -3 , 9

If $a = 4$, $\triangle = 64$, $r = \frac{1}{2}$ or $^{-3}/_{2}$ giving sequence 4, 2, 1 and 4, -6 , 9

If $a = 7$, $\triangle = 49$, $r = -1$ or 0 giving sequence 7, -7, 7 But 7, 0, 0 is not a geom seq.

If $a = 9$, $\triangle = 9$, $r = ^{-1}/_{3}$ or $^{-2}/_{3}$ giving sequence 9,-3, 1 and 9, -6 ,4

so there are 7 such sequences.

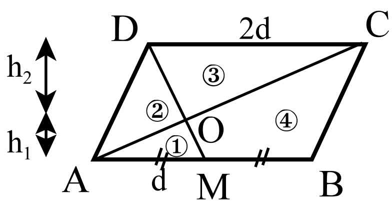
[Marker]

3. Let the areas of the regions be ①,②,③ and ④ smallest to largest

② + ③ = ① + ④ = ½ A

② = ½ A -③

④ = ½ A - ①



Notice that $\triangle DOC$ is similar to $\triangle ADM$ but is an enlargement scale factor 2.

Let $AM = d$ thus $DC = 2d$.

Let altitudes be h_1 and h_2, so $h_2 = 2h_1$. And ③ = 4 ①

$A = 2d(h_1 + h_2)$ $\triangle AOM$ ① = $\frac{1}{2}\, d\,.h_1$

$A = 2d\,(3h_1)$ ① = $^{1}/_{12}$ A

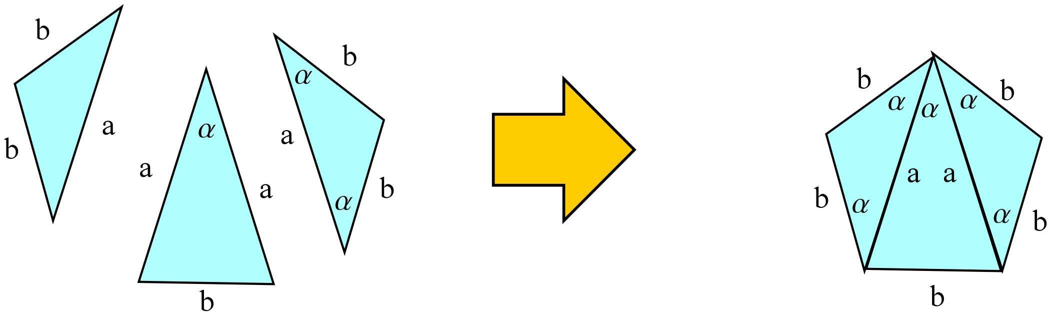
$^{1}/_{6}\, A = d\,.h_1$ hence ③ = $^{1}/_{3}$ A

and ② = $^{1}/_{6}$ A and ④ = $^{5}/_{12}$ A

[John Ong]

4. In second isosceles triangle we can see $\cos\alpha = \frac{1}{2} a / b$ so $2\cos\alpha = {}^{a}/_{b}$

Now draw another triangle with base angles α as shown and rotate as shown



now assemble to make a regular pentagon.

from above diagram $3\alpha = 108^{o}$ (int angle reg pentagon) ${}^{a}/_{b} = 2\cos 36^{o} = 1.618$ the golden ratio.
$\alpha = 36^{o}$

[Jimmy Yuan]

5. $$abc = x^2y^2 + 1 + \frac{1}{x^2y^2} + 1 + \frac{1}{x^2} + x^2 + \frac{1}{y^2} + y^2$$

$$a^2 = x^2 + 2 + \frac{1}{x^2} \quad b^2 = y^2 + 2 + \frac{1}{y^2} \quad c^2 = x^2y^2 + 2 + \frac{1}{x^2y^2}$$

$$a^2 + b^2 + c^2 - abc = x^2 + \frac{1}{x^2} + y^2 + \frac{1}{y^2} + x^2y^2 + 6 + \frac{1}{x^2y^2} - (x^2y^2 + \frac{1}{x^2y^2} + 2 + \frac{1}{x^2} + x^2 + \frac{1}{y^2} + y^2)$$

$$= 4$$

A constant as required. [Ashley Rankin]

6. Let $x = \sqrt{a}$ and $y = \sqrt{b}$ so $x^3 + y^3 = 32$ - ① and $x^2y + y^2x = 31$ - ②

① + ② gives $x^3 + x^2y + y^2x + y^3 = 63$

Also $(x + y)^3 = x^3 + 3x^2y + 3y^2x + y^3$
$= x^3 + x^2y + y^2x + y^3 + 2(x^2y + y^2x)$
$= 63 + 2(31)$
$= 125$

Thus $x + y = 5$ giving us $y = 5 - x$, sub into ① and solve

$x^3 + (5 - x)^3 = 32$
$x^3 + 125 - 75x + 15x^2 - x^3 = 32$
$15x^2 - 75x + 125 = 32$
$15x^2 - 75x + 92 = 0$
$5x^2 - 25x + 31 = 0$
$x = 2.724$ or 2.276
$y = 2.276$ or 2.724

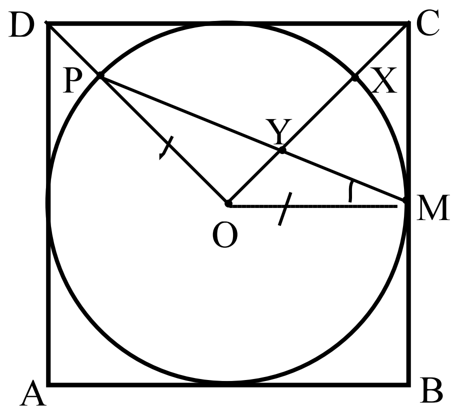
From above $a = x^2$ and $b = y^2$

$a = 7.42$ or 5.180
$b = 5.180$ or 7.42

so two solutions (a,b) are (7.42, 5.180) or (5.180 , 7.42)

[Tim Wang]

7. Let radius of circle = r



$\angle$ DOC = 90º since diagonals of square
$\angle$POM =135º since $\angle$COM =45º
$\angle$PMO =22.5º since PMO is isosceles
$\angle$CMY =67.5º since $\angle$CMO is 90º
$\angle$CYM = 67.5º sum of angles in $\triangle$CYM

Thus $\triangle$CYM is isosceles with CY = CM = r

hence OY + YX = r and YX + XC = r

thus OY = XC [Matt Ogden]

8. If the third person is first to have a duplicate birthday its probability is

$$1\times\frac{365}{366}\times\frac{2}{366}$$ (A choice of two dates for duplicate)

the fourth person is first to have a duplicate birthday its probability is

$$1\times\frac{365}{366}\times\frac{364}{366}\times\frac{3}{366}$$ (A choice of three dates for duplicate)

the fifth person is first to have a duplicate birthday its probability is

$$1\times\frac{(366-1)}{366}\times\frac{(366-2)}{366}\times\frac{(366-3)}{366}\times\frac{4}{366}$$ (A choice of four dates for duplicate)

the n th person is first to have a duplicate birthday its probability is

$$1\times\frac{(366-1)(366-2).....(366-(n-2))(n-1)}{366^{n-1}}$$

the n +1 th person is first to have a duplicate birthday its probability is

$$1\times\frac{(366-1)(366-2).....(366-(n-2))(366-(n-1))(n)}{366^{n}}$$

we require P(nth) > P(n+1th)

$$1\times\frac{(366-1)(366-2).....(366-(n-2))(n-1)}{366^{n-1}} > 1\times\frac{(366-1)(366-2).....(366-(n-2))(366-(n-1))(n)}{366^{n}}$$

$$\frac{(n-1)}{366^{n-1}} > \frac{(366-(n-1))(n)}{366^{n}}$$

$$366\times(n-1) > (367 - n)\,n$$

$$366n - 366 > 367n - n^2$$

$$n^2 - n > 366$$

$$n > ½ + \sqrt{(366.25)}$$

$$n > 19.64$$

Thus should join at the twentieth position. [Marker]

9.

$$(\ln x)^2 - 2{\cdot}5(\ln x)(\ln(4x-5)) + (\ln(4x-5))^2 = 0$$
$$(\ln(4x-5))^2 - 2{\cdot}5(\ln x)(\ln(4x-5) + 1.25^2(\ln x)^2 = {}^9/_{16}(\ln x)^2$$
$$(\ln(4x-5) - {}^5/_4 \ln x)^2 = {}^9/_{16}(\ln x)^2$$
$$\ln(4x-5) - {}^5/_4 \ln x) = \pm\, {}^3/_4 (\ln x)$$

For $\ln(4x-5) - {}^5/_4 \ln x = {}^3/_4 \ln x$

$$\ln(4x-5) - 2\ln x = 0$$
$$\ln\left(\frac{4x-5}{x^2}\right) = 0$$
$$\frac{4x-5}{x^2} = 1$$
$$x^2 - 4x + 5 = 0$$

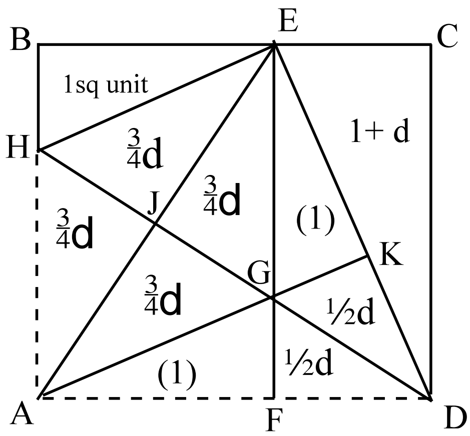
No Real Solutions

For $\ln(4x-5) - {}^5/_4 \ln x = -\,{}^3/_4 \ln x$

$$\ln(4x-5) - {}^1/_2 \ln x = 0$$
$$\ln\left(\frac{4x-5}{\sqrt{x}}\right) = 0$$
$$\left(\frac{4x-5}{\sqrt{x}}\right) = 1$$
$$4x - \sqrt{x} - 5 = 0$$
$$(4\sqrt{x} - 5)(\sqrt{x} + 1) = 0$$
$$\sqrt{x} = \frac{5}{4} \quad or \quad \sqrt{x} = -1$$
$$x = \frac{25}{16} \quad or \quad x = 1$$

At x = 1, $\ln(4x-5)$ is not real, so $x \neq 0$ therefore $x = \frac{25}{16}$ only. [Jimmy Yuan]

10.



Draw EF, parallel to AB.
Draw AK, passing through G (the point of intersection of EF and DH). K is on ED.
Draw AE, intersecting DH at J

△BEH has an area of 1

so if △ECD has an area of 1 + d

then △DEH has an area of 1+ 2d

In HEGA, there is symmetry about AE and HG, hence ABCD is divided into 10 right angled triangles.

<u>Also</u> △EGK =△AGF=△BEH (area =1)

△CED=△EFD ⇒ △GKD =△GFD = ½ d

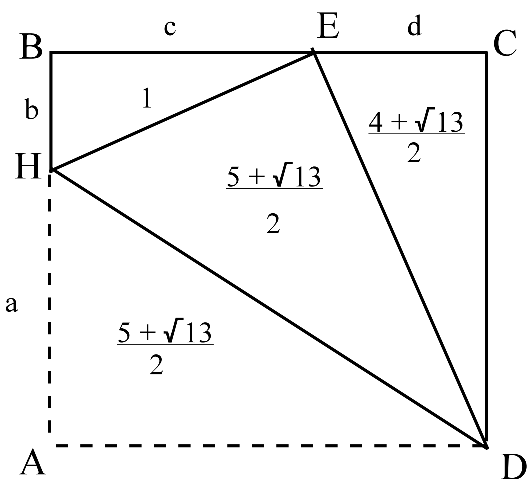
⇒4 △'s in rhombus AHEG have area ¾d

from pairs of similar △'s $\dfrac{\Delta \text{CED}}{\Delta \text{BEH}} = \dfrac{DE^2}{EH^2} = \dfrac{\Delta JED}{\Delta JEH}$

$$\frac{1+d}{1} = \frac{1+\frac{5}{4}d}{\frac{3}{4}d}$$
$$3d^2 - 2d - 4 = 0$$
$$d = \frac{1+\sqrt{13}}{3}$$

so areas of triangles are $1, \dfrac{4+\sqrt{13}}{3}, \dfrac{5+\sqrt{13}}{3}$ $(1, 1{\cdot}2535, 4{\cdot}070)$

To find the dimensions, label the diagram as below with areas.



Using areas $bc = 2$ - (1)

$d(a + b) = 5.070$ -(2)

$a(c + d) = 8.141$ -(3)

$(a + b)(a + b) = 11.678$ -(4)

$\Rightarrow a(c + d) + bc + bd = 11.678$

$bd = 1.535$

from (2) $ad + bd = 5.070$ So $ad = 3.535$

$\Rightarrow {}^{bd}/_{ad} = {}^{1.535}/_{3.535}$ $b = 0.434a$

$c = {}^{2}/_{b}$ So $c = {}^{4.608}/_{a}$

$a^2 = b^2 + c^2$

$a^2 = (0.434a)^2 + ({}^{4.608}/_{a})^2$

$a^4 = 26.163$

$a = 2.262$, $b = 0.982$, $c = 2.037$ and $d = 1.563$

The dimensions of the rectangle are $(a + b)$ by $(c + d)$

that is 3.244 by 3.60 units

[Marker]