CASIO SENIOR MATHEMATICS COMPETITION 2008

Note: ***It is not possible to provide all possible methods students will use to solve these questions. If an answer is correct award full marks.***

1. Find the value of $2008^2 - 2007^2 + 2006^2 - 2005^2 + \dots + 4^2 - 3^2 + 2^2 - 1^2$ [4]

Grouping in pairs $= (2008^2 - 2007^2) + (2006^2 - 2005^2) + \dots + (4^2 - 3^2) + (2^2 - 1^2)$ ✔

$= 4015\times1 + 4011\times1 + \dots + 7\times1 + 3\times1$ ✔

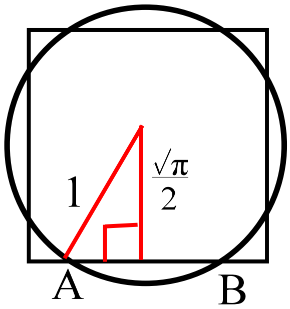
which is an arithmetic sequence with 1004 terms

$= \frac{1004}{2}\{3 + 4015\}$ ✔

$= 2017036$ ✔

2. A circle, of radius 1 cm, and a square are concentric.

If the area of circle and the square are equal, what is the length of the line segment AB?



The area of the circle and square are both $\pi\ cm^2$

$\Rightarrow$ *side length of square* $= \sqrt{\pi}$ *cm* ✔

$\therefore$ *by Pythagoras* $\left(\frac{AB}{2}\right)^2 + \left(\frac{\sqrt{\pi}}{2}\right)^2 = 1^2$ ✔

$\Rightarrow AB = \sqrt{(4 - \pi)}$ cm

$= 0.927$ cm ✔

3. Let f satisfy the functional equation $2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 140$

Find $f(3)$

When $x=3$ $\quad 2f(3) + 3f(35) = 440$ (1) ✔

When $x = 35$ $\quad 2f(35) + 3f(^{99}/_{33}) = 3640$ (2) ✔

(1) × 2 $\quad 4f(3) + 6f(35) = 880$ (3)

(2) × 3 $\quad 9f(3) + 6f(35) = 10920$ (4) ✔

(4) - (3) $\quad 5f(3) = 10040$

$f(3) = 2008$ ✔✔

4. Three judges for a talent quest have to vote publically on the three performers A, B and C, ranking their order of preference.

What is the probability that the judges vote so that two of them agree in their order of preference while the third differs?

Solution 1: *possible ways that are possible*

AAD , ADA, and DAA where A = Agree and D is differ ✔

A judge with no restrictions has 6 different ways to place the contestants in order.
A judge with the same order as a previous judge has only 1 order possible.
A judge with a different order has 5 possible orders. ✔

Hence Number of ways so that two agree is $3 \times (6 \times 1 \times 5) = 90$ *ways* ✔
Number of ways judges can select with no restriction is $6 \times 6 \times 6 = 216$

Probability(that only two of the three agree) = $^{90}/_{216}$ $(= ^{5}/_{18} = 0.2777..)$ ✔

Solution 2: *The total number of ways the judges can select the order is* $6 \times 6 \times 6 = 216$ ✔
The number of ways so that all selections are different is $6 \times 5 \times 4 = 120$ ✔
The number of ways so that they are all the same is $6 \times 1 \times 1 = 6$ ✔
Hence the number of ways so that two judges agree on the order is $216 - (120+6) = 90$

Probability(that only two of the three agree) = $^{90}/_{216}$ $(= ^{5}/_{18} = 0.2777..)$ ✔

5. Find the smallest positive integer *n* such that $\sqrt{n} - \sqrt{(n-1)} < \dfrac{1}{2008}$

The left hand side can be rewritten as

$$\left(\sqrt{n} - \sqrt{(n-1)}\right)\left(\frac{\sqrt{n} + \sqrt{(n-1)})}{\sqrt{n} + \sqrt{(n-1)}}\right)$$ ✔

$$= \left(\frac{1}{\sqrt{n} + \sqrt{(n-1)}}\right)$$ ✔

⇒ *the given inequality becomes* $\dfrac{1}{\sqrt{n} + \sqrt{n-1}} < \dfrac{1}{2008}$

⇒ $\sqrt{n} + \sqrt{(n-1)} > 2008$ ✔

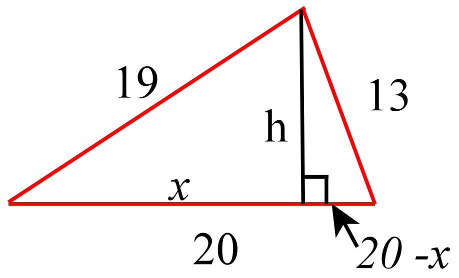
but $\sqrt{1,008,016} = 1004$

so $\sqrt{1,008,016} + \sqrt{1,008,015} < 2008$

And $\sqrt{1,008,017} + \sqrt{1,008,016} > 2008$

thus, the smallest value of n is 1,008,017 ✔

6. A triangle has sides of 19 cm, 20 cm and 13 cm. Find the length of the altitude perpendicular to the side of length 20 cm which divides the triangle into two equal area.



Draw the altitude from the vertex opposite the 20 cm side and label as shown.

Using Pythagoras

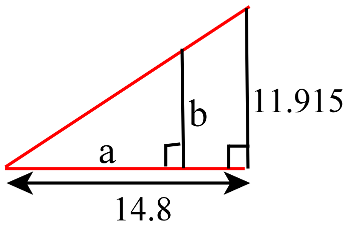
$h^2 = 19^2 - x^2 = 13^2 - (20 - x)^2$

solving gives

$x = 14.8$ and ✔✔

$h = 11.915$ cm ✔

Area $= 119.15$ cm^2 ✔



Using the left side triangle, with the triangle that has at least half the area of 119.5 cm^2

Using similar triangles

$$\frac{b}{a} = \frac{11.915}{14.8}$$

So $a = 1.242\,b$ ✔

Area of required triangle $\tfrac{1}{2}ab = \tfrac{1}{2}(119.15)$

$1.242\, b^2 = 119.15$

$b = 9.795$ cm

the required altitude is 9.795 cm ✔

Alternative .

Find angle θ between side 19 and 20 cm. (Use cosine rule)

$\theta = 38.8357^\circ$

Find Area using $\tfrac{1}{2}\, 19\times 20 \sin 38.8357^\circ = 119.14697$

Use $a = b/\tan 38.8357^\circ$

So $\tfrac{1}{2}\, b^2 / \tan 38.8357^\circ = \tfrac{1}{2}(119.14697)$

$b^2 = 119.14697 \tan 38.8357^\circ$

$b = 9.79382$

the required altitude is 9.794 cm

7. A cubic equation has three roots which are perfect squares such that $a^2 + b^2 = c^2$, where a^2, b^2 and c^2 are the three roots.
If the equation is $x^3 + px^2 + qx + r = 0$, find the relationship that links p, q and r .

$(x - a^2)(x - b^2)(x - c^2) = x^3 + px^2 + qx + r$

$x^3 - (a^2+b^2+c^2)\, x^2 + (a^2 b^2 + (a^2 + b^2)\, c^2)\,x - a^2 b^2 c^2$

$a^2 + b^2 + c^2 = 2c^2 = -p$ (1)

$a^2 b^2 + (a^2 + b^2)\, c^2 = q$ (2) ✔✔

$a^2 b^2 c^2 = -r$ (3)

So $c^2 = -p/_2$ (5) ✔

Putting $a^2 + b^2 = c^2$ *into (2)*

so $a^2 b^2 + c^4 = q$ (4)

From (3) $a^2 b^2 = -r / c^2$ *and using* $c^2 = -p/_2$

gives $a^2 b^2 = 2r/p$ (6) ✔✔

Since $a^2 b^2 + c^4 = q$ (4)

Substituting (5) and (6) into (4) **Gives** $^{2r}/_p + (-p/_2)^2 = q$

Or $4pqr = p^3 + 8r$ ✔

8. Draw six cards from a standard pack of 52 playing cards without replacement.
How many distinct ways can you choose these six cards so that all the following conditions are met.

i) The first card drawn is a spade,
ii) the second card drawn is also a spade,
iii) the third card drawn is a club,
iv) the fourth card drawn is a diamond,
v) the fifth card drawn is a red card (either heart or diamond),
and
vi) the last card drawn is an ace.

Work from the last condition back, last card is an Ace so we need to examine 4 cases.
Starting with Ace of Spades - A♠

i) The first card drawn is a spade, - 12 ways since Ace used
ii) the second card drawn is also a spade,- 11 ways since two cards have been used
iii) the third card drawn is a club, - 13 ways
iv) the fourth card drawn is a diamond, - 13 ways
v) the fifth card drawn is a red card (either heart or diamond), - 25 ways since a red card has been used
So 12 ×11 ×13 ×13 ×25 =557,700 ways ✔✔✔

Starting with Ace of Clubs - A♣ and using similar logic
♠ × ♠ × ♣ × ♦ ×(♦or ♥)
13 × 12 ×12 ×13 ×25 =608,400 ways ✔

Starting with Ace of Hearts - A♥ and using similar logic
♠ × ♠ × ♣ × ♦ ×(♦or ♥)
13 ×12 ×13 ×13 ×24 = 632,736 ways ✔

Starting with Ace of Diamonds - A♦ and using similar logic
♠ × ♠ × ♣ × ♦ ×(♦or ♥)
13 × 12 × 13 ×12 × 24 = 584, 064 ways ✔

Total number of ways to satisfy the six conditions is 2,382,900 ways ✔

9. By definition, Euler's number (e) is defined as $\lim_{n\to\infty}(1+\frac{1}{n})^n$ which is approximately equal to 2.7128182846.

What is the value of $\lim_{n\to\infty}(1+\frac{2}{n})^n$.

Derive your solution without using calculus, only algebra.

Let $x = \lim_{n\to\infty}(1+\frac{2}{n})^n$ *and since* $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$

$= \lim_{n\to\infty}\left(\frac{n+2}{n}\right)^n$ ✔ $= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$

dividing x by e gives $\frac{\ }{e} = \lim_{n\to\infty}(\frac{n+2}{n+1})^n$

$= \lim_{n\to\infty}\left(1 + \frac{1}{n+1}\right)^n$ ✔

Let u = n + 1

$$\Rightarrow \frac{x}{e} = \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u-1}$$ ✔

But as increase without bound, u-1 is equivalent to u ✔

So $$\frac{x}{e} = \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{u}$$

$$\frac{x}{e} = e$$ ✔

$$x = e^2$$ ✔

10. Find the limiting value of the series $\frac{3}{10}+\frac{21}{100}+\frac{117}{1000}+\frac{609}{10000}+\ldots$

the denominator is the sequence $10, 10^2, 10^3, 10^4$ *ie* 10^n ✔

the numerator is in the form $5^1 - 2^1, 5^2 - 2^2, 5^3 - 2^3$, *ie* $5^n - 2^n$ ✔✔

the series becomes $$\sum_{n=1}^{\infty}\frac{5^n-2^n}{10^n}$$

$$=\sum_{n=1}^{\infty}\frac{5^n}{10^n}-\frac{2^n}{10^n}$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n-\left(\frac{1}{5}\right)^n$$ ✔ *the difference between two infinite geometric sequences.*

$$=\frac{\frac{1}{2}}{1-\frac{1}{2}}-\frac{\frac{1}{5}}{1-\frac{1}{5}}$$

$$=1-\frac{1}{4}$$

$$=\frac{3}{4}$$ ✔

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